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Citation to Publisher Version:

Schorr, R. Y., Epstein, Y. M., Warner, L. B. & Arias, C. C. (2010-b). Mathematical truth and social consequences: The intersection of affect and cognition in a middle school classroom. *Mediterranean Journal for Research in Mathematics Education,* 9 (1), 107-134.

Citation to this Version:

Schorr, R. Y., Epstein, Y. M., Warner, L. B. & Arias, C. C. (2010-b). Mathematical truth and social consequences: The intersection of affect and cognition in a middle school classroom. Retrieved from http://hdl.handle.net/20.500.12164/114

Mediterranean Journal for
Research in Mathematics Education
Vol. 9, 1, 107 – 134, 2010

Mathematical Truth and *Social* Consequences: The Intersection of Affect and Cognition in a Middle School Classroom

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ABSTRACT: *We focus on several groups of eighth grade students in an attempt to deepen our understanding of when, how, and why middle grade students engage in conceptually challenging mathematics. As part of our analysis, we have formulated the theoretical notion of engagment structures, which is a behavioral/affective/social constellation including characteristic patterns of behavior, indicative of affective pathways and models (structures) that have important cognitive interpretations and implications by the students. We report that students may be willing to abandon arguing for what they know are mathematical truths in order to avoid appearing weak or wrong in front of their peers, and this appears to be linked to the depth of their understanding and their social positioning within their groups.*

Key words: Affect; engagement; problem solving; mathematical understanding, urban education.

INTRODUCTION:

The study reported in this paper reinforces the growing body of research that emphasizes the importance of affect in mathematical learning, and forms the foundation for one of the major research initiatives of the Rutgers University based MetroMath

Center for Learning and Teaching funded by the National Science Foundation (NSF)[1]. One main purpose of our work on the *Affect Study* is to document the development of what we refer to as *mathematically powerful affect* as it relates to student engagement in *conceptually challenging mathematics* (these terms will be described in greater detail below). In particular, we focus on urban students whose mathematical underachievement may belie their actual capabilities (Schorr, Warner, Gearhart, & Samuels, 2007).

The study reported herein was designed to deepen our understanding of when, how, and why students engage in conceptually challenging mathematics. In the sections that follow, we will describe our theoretical perspectives and provide a brief description and analysis of a group of students solving a mathematical problem in an urban middle school classroom in the largest city in New Jersey, USA.

As a point of clarification, urban, as it is used in this research, is meant to convey a large array of issues often associated with large cities in the USA (and elsewhere) including, but not limited to high population density, higher rates of unemployment (than in the surrounding suburban communities) and large minority populations. A fundamental premise of our research is to identify focus areas where specific characteristics of such environments emerge as major current or potential influences on mathematics learning.

We focus, in particular, on an interesting observation: that students may be willing to abandon arguing for what they know are mathematical truths in order to avoid appearing weak or wrong in front of their peers (see Epstein et al., 2007; Goldin et al., 2007; Schorr et al., 2010). In this particular case, Dana, an eighth grade girl, defended a solution that she knew might be incorrect when another student, Shay, an eighth grade boy, pointed out an error in her work in front of a few of her classmates (including the three other students that she had worked with to solve the problem). Immediately prior to this, Dana had been very interested in understanding the concept; indeed, she was actively trying to understand where and how she might have made an error in her solution (after noticing that Shay's group had a different solution). After Shay's semi-public criticism, she abandoned her quest for mathematical truth and sought to avoid any signs of weakness; in effect, she sought to avoid "losing face". We speculate that

[1] Grant No. ESI-0333753 (MetroMath: The Center for Mathematics in America's Cities). Additional support for this work comes from the Newark Public School Systemic Initiative in Mathematics (Local Systemic Change Grant No. ESI-0138806). Any opinions, findings, and conclusions or recommendations are those of the authors and do not necessarily reflect the views of the NSF or Rutgers University or any of the schools or districts reported on herein.

she had a number of options, one of which involved acknowledging a mathematical mistake in her work. However, she chose to defend her incorrect solution, in a rather public manner. She noted, in an interview that occurred just after the session, that she does not like it "…when people try to prove *me* wrong", especially in front of her peers. Her response is consistent with the findings of Dance (2002), Anderson (2000), and Devine, (1996), who note that young people are often hypersensitive to situations in which their emotional safety, status, or wellbeing may be challenged.

In this paper, we will attempt to make sense of this and other similar behaviors using several different lenses: cognitive, affective, and social.

THEORETICAL FRAMEWORK

We have used several terms that are in need of greater clarification. To begin, by *conceptually challenging mathematics,* we mean mathematical concepts that may not be offered to students because of their perceived difficulty level, or because of the cognitive or conceptual hurdles that they may pose to students. Such topics may include rational number concepts, algebraic reasoning, etc. We also mean learning and teaching for conceptual understanding, including making connections, constructing and exploring representations, mathematical abstraction, modeling, defending and justifying solutions, and non-routine problem solving. Our research, and the research of others, suggests that effective classroom activity surrounding conceptually challenging mathematics is likely to involve mathematical discussion, exploration, individual students expressing their own ideas, disagreements, "wrong answers" and "blind alleys", as well as fruitful suggestions, challenges, and thoughtful questions posed not only by the teacher but by students to each other (Lesh & Doerr, 2004; Schoenfeld, 1992; Schorr & Goldin, 2008). As this occurs, students and teachers are likely to experience a variety of emotions including, but not limited to– vulnerability, curiosity, puzzlement, bewilderment, confusion, annoyance, anger, fear, a sense of threat, defensiveness, suspicion, pleasure, etc. (DeBellis & Goldin, 2006; Goldin, 2000, 2007; McLeod, 1992, 1994; Schorr and Goldin, 2008). Many of these experiences may involve some degree of emotional risk-taking and related changes in affect and have important cognitive and social consequences.

Understanding the emotional aspect of mathematical problem solving requires a review of the groundbreaking work done over the past two decades by Gerald Goldin (see for example, DeBellis & Goldin 1997, 1999, 2006; Goldin, 1988, 2000, 2002, 2007). His work builds upon an approach to systems of representation (Goldin 1998; Goldin & Kaput, 1996) and is inspired by other researchers (including Dai & Sternberg, 2004; Evans, 2000; Hannula, 2002, 2004; Leder, Pehkonen, & Törner, 2002; Lesh, Hamilton,

& Kaput, 2007; Malmivuori, 2001; McLeod, 1992, 1994; Zan, Brown, Evans, & Hannula, 2006). This work lays the foundation for what we refer to as mathematical powerful affect, which will be described below.

Powerful mathematical affect in students is not the same thing as positive affect (Goldin, 2000, 2007; McLeod, 1992, 1994). It is not focused exclusively on positive emotional feelings, such as curiosity, interest, and satisfaction. Rather, it also may include feelings such as impasse, frustration, and disappointment. The main difference is that these feelings lead to mathematical engagement, persistence, problem-solving success, and achievement rather than feelings of humiliation, embarrassment or failure. In this study, we approach the challenging problem of characterizing mathematically powerful affect, especially as it relates to the social and cognitive aspects of students solving problems in urban middle school mathematics classrooms.

As we have already suggested, student actions or reactions to a particular stimulus complex in mathematics classrooms are not isolated or decontextualized, but rather the result of many factors involving their social, cultural, contextual, cognitive, and affective experiences. Consequently, we draw upon several interdisciplinary perspectives to guide our research and help us to analyze how the particular cultural and contextual aspects of the urban environments in which the students live affect them, their interactions with their peers and teachers, and of course, their mathematics learning. We draw upon the research base in socio-cultural theory (e.g., Cobb, & Yackel, 1998; Cole, 1996; Wertsch, 1985) and situated learning theory (e.g., Anderson, Reder, & Simon, 1996; Brown, Collins, & Duguid, 1989), cognitive science (e.g., Greeno, Collins, & Resnick, 1995), and mathematics education (e.g., Lesh, Hamilton, & Kaput, 2007). In addition, we attend to the literature base focusing on the specific challenges faced by urban students (e.g., Anderson, 1999; Dance, 2002).

In order to analyze the cognitive aspect of our findings, we draw upon the Pirie-Kieren (1994) model. According to this model, one can view the growth of mathematical understanding via a number of layers through which students move both forward and backward. The Pirie-Kieren model describes several different categories that are used in characterizing the growth of understanding, while observing understanding as a dynamic process and not as a single or multi-valued acquisition (nor as a linear combination of knowledge categories). Pirie and Kieren illustrate eight potential layers, or distinct modes, within the growth of understanding for a specific person, on any specific topic. Two examples of layers that are discussed in our results section below are “image making” and “image having”.

When the learner is “doing something” to get the idea of what the concept is, he/she is working in the *image-making* layer. A person working in the image-making layer is "tied to the action" or "tied to the doing". "At the level of image-having, a person can use a mental construct about a topic without having to do the particular activities which brought it about" (Pirie & Kieren, 1994, p. 66). One has an image and has reached a "don't need" boundary where he/she is no longer "tied to the action or doing". We use this model in order to characterize two students' understanding, especially as it relates to observed shifts in affect.

Central to our analysis of affect, we draw upon Cohen's (1978) capacity model of attention. According to this model, attention is a limited commodity that needs to be allocated amongst tasks. Circumstances demanding more attention get more attention, while those needing less, get less. Consequently, tasks or occurrences addressing high priority goals are given greater attention than tasks associated with lower priority goals. Accordingly, one important stimulus for many youngsters in urban environments is *danger* that can arise from an insult (tacitly or explicitly expressed) by another, an act that makes one look wrong or foolish, or lose "face" (Anderson, 2000; Dance, 2002; Devine, 1996; Fine, 1991). On the other hand, "Face-saving behaviors and the avoidance of face loss interactions enhance smooth relations among group members and help minimize disruptions to the social order" (Zane & Yeh, 2002, p. 126). Further, according to Anderson (2000), in his book *Code of the Street*, "life in public often features an intense competition for scarce social goods in which 'winners' totally dominate 'losers' and in which losing can be a fate worse than death" (p. 37). He further notes that an important aspect of this "code" is to avoid being perceived as weak or a "loser". We conjecture, based upon our analysis and data, that many of the students we are studying are hyper-vigilant for incidents in which their honor or "face" will be challenged during discussions about mathematical ideas. Thus, they allocate a significant portion of their attention (per Cohen's model) to monitoring for these stimuli at the expense of allocating this attention to the mathematical ideas involved in the task that they are working on.

There is no doubt that most middle school students, whether living in urban, suburban, or rural environments, have a variety of issues and concerns that compete for their attention. In particular, social issues such as peer group acceptance are high priority goals. Consequently, actions instrumental to attaining acceptance are likely to be allocated a large share of any student's attention. We are not claiming that these issues are unique to urban students, nor are we claiming that they only relate to young people in general. We do know, however, that in interviews conducted during our study, students often mentioned these issues.

In addition to the social issues discussed above, the middle school years (approximately ages 12-15, or grades six through eight) place unique demands on all students. Indeed

Eccles and Midgley (1989) note that middle school students *need* an environment that provides a 'zone of comfort'. In our own work, we also note the importance of providing students with what we term an *emotionally safe environment*. In such an environment, the students are free to question ideas, and openly discuss (mis)understandings without risk or fear of embarrassment or humiliation. Oftentimes, students in such an environment may work in small groups in which they are asked to engage in mathematical discourse that includes efforts to prove and justify contentions to peers and the teacher. Implicit in this discourse is a socio-mathematical norm that permits and even encourages students to challenge the ideas put forth by their peers (Cobb, Wood, & Yackel, 1993; Franke, Kazemi, & Battey, 2007; Warner, Schorr & Davis, 2009).

There is little doubt that the teacher is of key importance in shaping the emotional safety of any classroom, and the corresponding discourse patterns of practice. This then directly relates to how and what students learn: "How teachers and students talk with one another in the social context of the classroom is critical to what students learn about mathematics and about themselves as doers of mathematics" (Franke et al., 2007, p. 230). Nonetheless, given an emotionally safe environment, *and* a teacher who actively seeks to instill classroom norms that encourage productive and meaningful mathematical exploration and discourse, different students will engage with mathematical problems in different ways and at different times. This in turn, ultimately impacts their overall experiences and understanding. No matter the classroom, there may well be students who, at some point or another, feel threatened or uncomfortable when their work is criticized—this in turn can lead to changes in their affect and the engagement structures that are evoked.

DeBellis and Goldin developed a framework in which essential affective structures include mathematical integrity, mathematical self-identity, and mathematical intimacy (1997, 1999). Integrity entails commitment to truth and understanding in mathematical activity, self-awareness at any point of the limitations of one's mathematical understanding, and willingness to work to increase or deepen understanding. Self-identity encodes one's personal sense of oneself in relation to mathematics—"who I am" as a doer or user or learner of mathematics. And mathematical intimacy pertains to structures of emotions, attitudes, beliefs and values associated with intense, engaged, and vulnerable interaction in doing mathematics, possibly characterizing one's personal relationship with mathematics. These structures—self-identity, integrity, and intimacy—are key features in the concept of an *engagement structure*, which we have formulated as part of our analysis. Engagement structures, as we use it in our work, refers to a recurring pattern, inferred from observing the classroom and interview tapes, that is a kind of *behavioral/affective/social constellation.* Included are characteristic patterns of behavior, indicative of affective pathways and models (structures) that have important cognitive interpretations and implications by the students. In our work, we also often use the term "archetypal" in conjunction with engagement structures. Archetypal is intended to suggest the idealized nature of the patterns, abstracted from the observed instances of their occurrence.

Central to this is the notion of a "cognitive interpretation" which borrows from the seminal idea of W.I. Thomas (1923), which he called "the definition of the situation." Thomas states that, "preliminary to any self-determined act of behavior there is always a stage of examination and deliberation which we may call the definition of the situation." Waller (1932) states that, "the definition of the situation is a process. It is the process in which the individual explores the behavior possibilities of a situation, marking out particularly the limitations which the situation imposes upon his behavior, with the final result that the individual forms an attitude toward the situation, or more exactly, in the situation" (p. 292). Thomas (1923) states: "If men define situations as real, they are real in their consequences." The definition of the situation sets a context for which affective structures are likely to become activated in a given situation. One part of a structure, the interpretations and implications, take the form of what we describe as hypothetical *self-talk* and associated affective responses.

Engagement structures involve several interacting components that are described below (see Goldin, Epstein, & Schorr, 2007 for a more complete description). They include the following:

(1) a characteristic pattern of behavior, beginning in response to particular circumstance in the social context, and ending in a somewhat characteristic behavioral outcome,

(2) a characteristic sequence of emotional feelings, or affective pathways,

(3) meanings that may be encoded by the emotional feelings that are generated,

(4) self-talk or inner speech, in response to and evocative of the person's emotional feelings,

(5) characteristic problem-solving strategies for decision-making,

(6) interactions with the individual's systems of beliefs and values,

(7) interactions with the individual's structures of self-identity, integrity, and intimacy,

(8) meta-affect, that includes feelings about feelings, feelings about cognition about feelings, and self-monitoring of affect, and

(9) expressions from which affect may be inferred that are socioculturally-dependent as well as idiosyncratic, which can serve some communicative function, including eye contact and facial expressions, "body language," interjections and exclamations, tears and laughter, blushing, etc.

Our contention is that engagement structures can lead to important day-by-day choices, with powerful consequences—positive or negative—for students and teachers. An example of this will be highlighted below; others are documented in Epstein et al., (2007), Goldin, Epstein, and Schorr (2007), Schorr, Warner, and Arias (2008), and Schorr, Warner, Epstein, and Arias (2010). Further, in observing the students described in this paper who were working in groups on a mathematical task, we sometimes notice what seem to be critical "choice points" or "branch points." As we refer to them here, branch points occur in engagement structures when someone can act (consciously or otherwise) in one way rather than another, thus experiencing one set of feelings rather than another and evoking one structure instead of another.

We have identified several different types of such structures**.** Certain structures contribute directly to mathematical engagement, while others, at times, impede it. All, however, have important cognitive consequences ranging from decreased learning opportunities to increased learning opportunities. We see most or all of these structures as present within individuals and becoming operative under given sets of circumstances. We have identified at least seven different types of structures that impact engagement ranging from extreme engagement, which we refer to as *I'm Really Into This*, akin to what Csikszentmihalyi (1990) describes as "flow" (complete immersion in a task or activity), to complete disengagement, where students do what Kohl (1994) describes as "not learning," which is tantamount to an "active, often ingenious, willful rejection of even the most compassionate and well-designed teaching." (p. 2).

For the purposes of this report, we focus on several of the sample types of structures described below.

- Check this Out (CTO): This structure entails the individual's realization that solving the mathematical problem can have a payoff – either immediately, or at some future point. The resulting motivation to engage mathematically can lead to (intrinsic) interest in the task itself, or heighten (extrinsic) interest in an external payoff.
- I'm Really Into This (IRIT): This structure involves an individual's deep concentration in the situation at hand—solving the mathematical problem. The concentration is generally so deep that it can result in the experience of "flow," with the mathematical activity becoming so intriguing that the student "tunes out" his or her surroundings.

- Get the Job Done (GTJD): This structure involves a person's sense of obligation to fulfill his part of a work "contract." Ultimate satisfaction comes from doing the work in accordance with the contract rather than enjoying the challenge of the task.
- Looking Good (LG, also know as pseudo engagement): This structure involves a person's need to avoid working on a task, either because the person may not feel able to solve it, or may simply not wish to solve it. In an effort to avoid confrontations with the teacher or other peers, the person makes an effort to look as though he or she is actually engaged.
- Stay Out of Trouble (SOOT): This structure involves the person's need to avoid commitments or interactions that may lead to trouble – either with peers, or with an authority figure. Aversion to risk supersedes the mathematical aspects of the task.
- Don't Disrespect Me (DDM): This structure involves the person's experience of a perceived challenge or threat to his or her well-being, status, dignity, or safety. Resistance to the challenge raises the conflict to a level above that of the original mathematical task. The need to maintain "face" supersedes the mathematical issues.
- Look How Smart I Am (LHSIA): This structure involves the person's desire to demonstrate to others that he knows something that the other person does not know (as well as he does). Sometimes the primary motive is to have others recognize the person's superior ability. Embedded within this structure, in times where there is a disagreement, is a desire to persuade the other that "I'm right and you are wrong" or at least that my way of thinking about the problem is better than yours.
- Let Me Teach You (LMTY): This structure involves the person's (tutor's) desire to teach another person (tutee) something that he knows that the other person does not know. When a person initiates such help, it can be met with a number of different responses. The tutee can resist the help feeling that the tutor is trying to show her up. Or, the tutee can be grateful for the help, learn something important, become interested in the problem and the tutor can feel good about helping and proud of himself. In other words, the direction this structure takes at branch points is unclear and can go in different directions. If the tutee resents the help, she can attempt to devalue the information as well as devaluing the tutor himself. If that happens, a possible response by the tutor is to try to show how smart he is.

Different structures vary in the degree of engagement that they recruit. The greatest engagement of the structures described above occurs in the I'm Really Into This structures, with a close second occurring in the "Check This Out" structure. Less intense engagement is seen in the "Get The Job Done" structure. All three of these are very different from the disengagement seen when students have no interest in the task, or when students feign engagement as in pseudo engagement. Also, it is possible for a structure to operate in the service of another structure. For example, if the overarching motivation for a student is to stay out of trouble, he may, at times work to get the job done so as to avoid sanctions by the teacher or peers who need his input to get the job done.

We can also conceive of dimensions along which these structures can be located. One possible dimension is a motivational dimension that goes from intrinsic to extrinsic motivation on either pole. An orthogonal dimension is an intensity dimension going from high to low. As part of our analysis, we also consider what psychologists may refer to as "figure" and "ground" (Rubin, 1921, 1958). Figure, as we use it here, refers to the primary focus of attention, whereas ground refers to that which is present in the background. In our work, we have found that, at times, the mathematics may be figure and other aspects of the context may be ground, and vice versa.

Our goal in this paper is to better understand the differing modes of engagement, how they impact learning, and how they unfold as students criticize each other's work.

METHODS

Subjects: The eighth grade classroom that is the focus of this research consisted of 20 students, 93% African American and 7% Hispanic. The school, classified as "low income," is in the largest city in the state of New Jersey. The class was homogeneous and designated as a 'low ability' class (which in this case means that it was the lowest in terms of standardized test scores at the eighth grade level in this school). The teacher always encouraged an "emotionally safe" learning environment for students, a necessary condition for inclusion in the larger study (see below).

Procedure: In the larger study, all classes including the one that is the subject of this paper, were observed in each of four "cycles," with each cycle spanning a period of two consecutive days. The first cycle occurred approximately one month into the school year and subsequent cycles occurred later on. Prior to the start of a cycle, an interview was conducted with the teacher to ascertain her plans for the lesson and what she expected to happen. A post-lesson interview (using a stimulated recall protocol) with the teacher and individual interviews with several pre-selected students designated by the teacher, took place after each cycle (for more details see Epstein et al., 2007). The two primary students described in this paper are Shay and Dana. Shay, a young male student was described by his teacher as being a bright and “street wise” student. Dana, a young female was described as being popular and eager to please, but also “tough”.

Classroom interactions for each of the classes were videotaped using three separate cameras (one stationary camera capturing whole-class interactions and two roving cameras, primarily capturing pre-selected students and their interactions) and all student and teacher interviews were videotaped using one camera. Transcripts were created from videotapes and student work was collected and digitized.

Analysis: A team of researchers from the fields of mathematics education, social psychology, mathematics, and cognitive science, reviewed and analyzed the data. All videos were viewed through four distinct, yet overlapping lenses: the mathematical (cognitive) lens; the affective lens, particularly with regard to engagement; teacher interventions (which are not addressed in this paper); and social interactions. In all cases, the structures that we have identified were created after observing the data, rather than ahead of time.

Development of structures: The stance we took while viewing the videotapes was shaped by the following question: How can we explain how the actions, interactions, and statements of the students we are observing make sense? Why would they say what they said and do what they did? As we tried to solve this puzzle, we began to develop the concept of engagement structures, which gave meaning to what we were observing.

For this paper, we analyze data from one class during the first cycle (about 43 minutes of instruction each day). The students were working in groups of three to five (groups formed according to the typical seating arrangements—no roles or assigned tasks were given to any students) on the following mathematical task[2], chosen by the teacher:

> *Farmer Joe has a cow named Bessie. He bought 100 feet of fencing. He needs you to help him create a rectangular fenced in space with the maximum area for Bessie to graze. Bullet 1: Draw a diagram with the length and the width to show the maximum area. Bullet 2: Explain how you found the maximum area. Bullet 3: How many poles would you have for this area if you need 1 pole every 5 feet?*

RESULTS & DISCUSSION

[2] The problem was adapted from Exemplars K-12 (2004) www.exemplars.com

We believe that cognition, affect, and social behavior are interconnected and impact one another. In presenting our results, we will interweave these strands showing how they impact, sometimes aid, and sometimes constrain mathematical understanding.

Overview of Shay: Shay began investigating the task alone while the other three members of his group worked together, spending a considerable amount of time talking about several non-mathematical issues (socially related). Thirteen minutes later, Shay announced to his group, in an apparent breakthrough, "You could do a lot of 'em, it could be like…wait…" We were struck by Shay's interjection of "wait". We wondered what emotional feelings were encoded in this interjection. This led us to posit the importance of self-talk or inner speech, in response to and evocative of the person's emotional feelings. In this case, the feeling was so profound that Shay externalized it. We further theorized that the power of this feeling created an intrinsic motivation to explore the mathematical ideas underlying this task.

Realizing that there could be many rectangles with a perimeter of 100 feet, he encouraged his group members to investigate some possibilities while he continued to quietly work, primarily alone, for most of the session. We hypothesized that during this time Shay was fascinated by the math, in what appeared to be a "Check This Out" structure, as will be described more completely below.

For the most part, Shay interacted with other members of his group only when the teacher asked questions and/or when the teacher encouraged the group to share ideas with each other. During this time, Shay expressed some difficulty in figuring out the length and width of potential rectangles. When the teacher walked over to his group, he told her that he couldn't find the height. When the teacher asked why not, he answered "Because, if you trying to find the height…you pick two heights and it won't work for the other one." Here we were struck by Shay's willingness to display certain vulnerability by expressing the difficulty he was experiencing in figuring out the problem. Goldin (2008) has referred to this sort of vulnerability as mathematical intimacy—the willingness to make oneself emotionally vulnerable—a necessary condition in trying to attain understanding when attempting to solve a mathematical problem. Such vulnerability would not be possible if Shay were trying to avoid losing face and making sure that others always saw him as smart and knowledgeable.

We suggest that during this timeframe, Shay was functioning in the *image making layer* of the Pirie-Kieren model in that he still needed to draw specific rectangles in order to create an image of what each potential solution could be, and was still tied to the action of drawing in order to figure out the length of the sides in any one specific case.

At the very end of the first session, Shay realized that the other group members were still not sure about how to find the area of these rectangles. Here, we note that Shay's group consisted of four people operating in two pairs—Frank and Lottie (a pair who worked together and supported each other) and Shay and Ray. Shay seemed concerned that Ray understand what was going on. He appeared uninterested in whether Lottie or Frank understood, however, despite the fact that they were part of the same group. At one point, Shay was invested in teaching and explaining to Ray what he (Shay) understood. At another point he said to his group mates, "You could do ummm…" He was interrupted by Frank who said "thirty plus". Shay ignored this and said to Ray "look watch this, look...hold on…twenty three…and then twenty seven, twenty seven. Twenty seven, twenty seven equals to a hundred. I could do another one." Ray responded by saying, "Let me see that." Shay was clearly engaged at this point but it is less clear to us which structure figures most prominently in explaining his actions. When he said, "look, watch this," one could argue that he was trying to demonstrate something in order to teach a concept to a group mate. One could also argue that Shay wanted to garner praise from Ray who became aware that Shay understood the problem. Both processes could have been operating simultaneously with one being more prominent than the other. Ray never provided praise to Shay about Shay's knowledge. What Ray did do was show interest by saying "Let me see that." Ray appeared to be reinforcing Shay's teaching rather than Shay's knowledge. We therefore speculate that it is more likely that Shay was motivated by a desire to teach Ray than to demonstrate how smart he was to Ray.

Shay did not try to help Lottie at all. When Shay was trying to teach Ray, Frank observed and wrinkled up his nose. Lottie said to Shay, "You guys might not be helping me right." Frank came to Lottie's defense and said, "See she (Lottie) got 24, 24, 26, 26." This attempt by Frank to support Lottie was taken as a challenge by Ray and by Shay. Ray responded by saying "We got that already" and Shay said, "That was the first one I did." Here, we infer that Shay reached a 'branch point', which was set in motion by Frank's statement. Shay and Ray seemed to take the comments as a challenge—an attempt to show that Shay was not the only one with good information. Shay responded by saying, "That was the first one I did." He may have been implying that the others took a long time to come up with something that he had figured out a while ago. This episode may help to illustrate the point that multiple structures can occur simultaneously and the one that rises to the forefront depends upon what the student chooses to give attention to at that point in time.

By the end of the first day's session, Shay said to his group members (with a hint of frustration), "That's what we been trying to do all this time"—that is, calculating the area of several different rectangles. He also realized that it would take a long time to construct all *possible* rectangles with a perimeter of 100 feet (using only whole numbers). He stated, "you can keep going but it can go all day". He now, for what appeared to be very practical reasons, assumed the role of leader in his group. He asked his group mates to provide him with supplies, such as a ruler and paper. He also directed them regarding how to proceed by saying, "Move, I know what to do, look…what number, look…we gonna start, I'm gonna do the lower numbers, you do the higher numbers…" The other three members of his group began working independently on the different rectangles that Shay had assigned them. Here, it appeared as if Shay spontaneously assigned himself the role of group leader in order to uncover the answer--by looking at as many rectangles as possible. For very pragmatic reasons, Shay needed his group members to complete the task by following his directions.

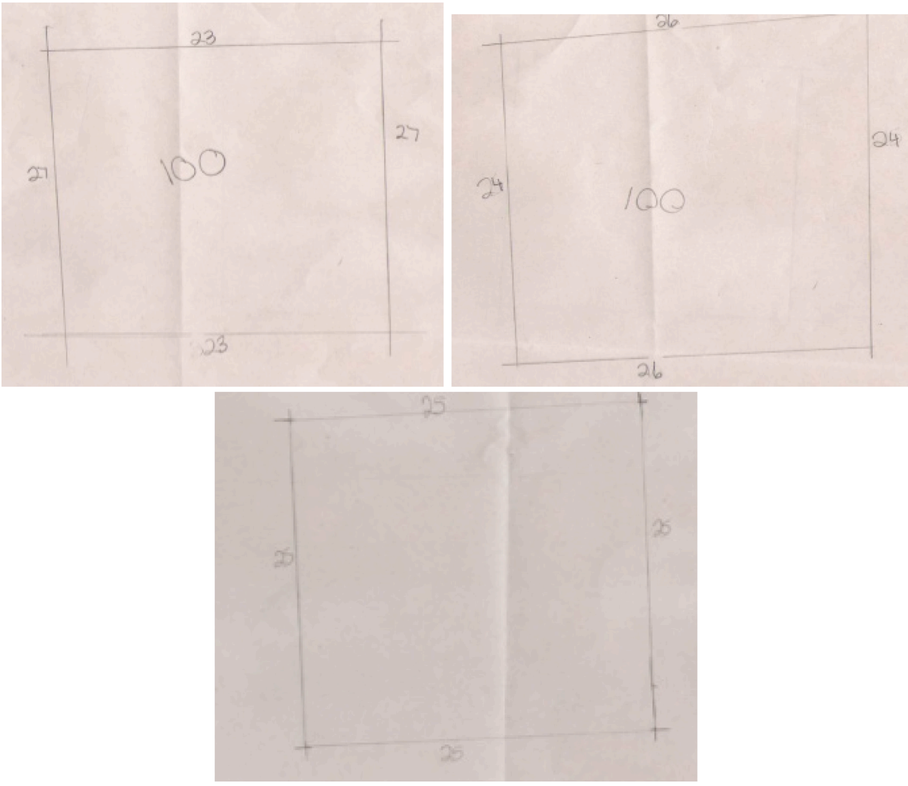

Figure 1: Shay's 23 x 27, 24 x 26, & 25 x 25 rectangles.

At first, Shay's choices of rectangles appeared to be random. He then proceeded to find the area of the rectangle with dimensions of 23 by 27 feet, 24 by 26 feet, and then 25 by 25 feet (see figure 1). As he continued to work, he began to develop strategies for building rectangles with a perimeter of 100. His method involved finding the width given a specific length. He added the side length to itself (for the length of two sides), subtracted the sum from 100, and divided the difference by two (to find the width). At this point, it appeared as if Shay had reached a "*don't need*" boundary and had an image of how he could construct rectangles with a set perimeter of 100. He was working in the *image having layer*, no longer tied to the action of drawing each rectangle to figure out the length and width. For any given length, he now had a method to find the width. He continued with this strategy until he came to the conclusion that the rectangle with the maximum area had dimensions of 26 by 24 (because he didn't consider the 25 by 25 square to be a rectangle). At this point, Shay also became invested in seeing to it that his group mates understood his method.

Based upon the data (including the video and student artifacts), we conclude that Shay spent most of the session engaged in a "Check This Out" structure, where the mathematics was figure, and the other aspects of the classroom context (social, for example) were ground. It was only after he became sure of his method that he monitored the progress of his group, and even then, the mathematics was central.

One could ask, why not consider Shay's involvement as an indication that he was motivated by a structure of "Get The Job Done" rather than "Check This Out?" After all, isn't requesting supplies and assigning roles to group mates an essential aspect of getting the job done? This highlights the idea that different structures can lead to similar outcomes. A teacher may ask students to find the solution to a problem and, at the end of a given period of time, the students may hand her a piece of paper containing that solution. But the structure leading to that outcome and the consequences of the process that unfolded on the path to that outcome may be very different. Shay's actions may have resulted in being able to hand the teacher a paper with the correct solution but this outcome is <u>incidental</u> to the structure that gave rise to that outcome. The motivation for Shay's actions was curiosity and a desire to explore an idea. His group mates, on the other hand, may have joined in with him in order to get the job done or to stay out of trouble by possibly offending him. This analysis suggests two points to keep in mind. First, it is not a good idea to infer the existence of a structure by examining the outcome. That outcome could be the result of several different motivating structures. Second, when a group of students work on a task, it is possible that each student may be operating under the influence of a structure different from the ones influencing his group mates at the same point in time.

Overview of Dana: On the first day of this problem solving exploration, Dana assumed (on her own initiative) the position of leading her group in the solution process. Essentially, the group discussed different potential perimeters for the fence. In the beginning of the session, Dana attempted to understand the task by asking questions and trying to figure out the meaning of area. Ultimately, under Dana's direction, they decided to use a 40-foot by 10-foot configuration for the dimensions of the fence. In order to calculate the area, Dana incorrectly assumed that she needed to multiply *lengths* [side a + side c] by *widths* [side b + side d] and she therefore calculated the area as 80 x 20 = 1600 [square feet]. Despite misconceptions (especially relating to area), Dana told her group mates what to do and when to do it.

After finding one rectangle that could satisfy the conditions of the first bullet (namely “draw a diagram to show the length and width, etc.”), Dana directed the group to consider another part of the problem task (related to the number of poles). She appeared to be motivated by a desire to “Get The Job Done”, since she appeared to be mainly interested in being sure that she had answered all parts of the question rather than formulating a more complete solution for each. In Dana’s case, the mathematics was ground, and monitoring her group’s activity was figure. She noted, “Yes. We need eight poles, so for the [third] bullet umm…two poles…two poles…” “So that’s…stop (to another group member) so that’s something that’s asking for bullet one, and bullet two, and bullet three. So, is everyone caught up yet (*addressing the other four members of her group*)? Ya got bullet three (*as she monitored their work*)?” Dana was concerned when someone in her group was off task—perhaps because it might get in the way of completing all of the parts of the problem. Like Shay, Dana made requests of group members to get her tools such as scrap paper and a calculator. Unlike Shay, Dana monitored each member’s progress. For example, a member of her group, Von, said to Dana, “Come on, I’m almost done.” Dana responded, “No you ain’t [sic].” It appeared that being an effective leader of her group was a very important feature of Dana’s self-identity.

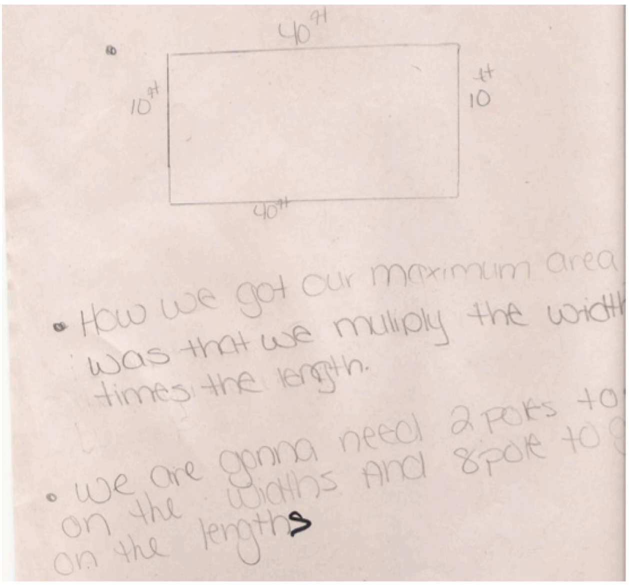

Figure 2: Dana’s group’s student work

We suggest that Dana did not have an image of how to create different rectangles with a constant perimeter when she finished the task. She used, as her final solution, the first rectangle (40 by 10) she constructed that had a perimeter of 100 (see figure 2). She didn't explore any other possibilities. She noted that in order to find the maximum area, she needed to "multiply length times width" without actually doing so. She never progressed past the *image making layer*, and unlike Shay, didn't reach a "*don't need" boundary* (where she was no longer tied to the action of drawing the rectangle.)

Dana and Shay's Interaction: On the following day, all groups of students were asked by the teacher to prepare a poster documenting their solution strategies. The purpose was to enable all students to walk around the room in order to review, comment, in writing (using small pieces of paper) and ask any questions that they had about another group's work. Both Shay and Dana recorded the solutions for their respective groups, and therefore had considerable ownership of the work.

As the different groups walked around the room, Dana seemed to be interested in seeing what others had chosen for the dimensions of the rectangular fences and the associated areas. As she looked at Shay's group's work, she noticed that the area was computed to be 624 [square feet] with the associated dimensions of 26 feet by 24 feet. The other members of her group appeared to be only casually interested in the differences. However, Dana appeared to be quite perplexed and intent on understanding the nature of the differences. She stated: "I don't understand. I want to know how they did that." It appeared to the research team that Dana's intense engagement with the task was guiding her actions.

> Dana: But I don't know how they got this, how they got maximum area…(inaudible)…I don't get that (*picking up a calculator and beginning to use it*). But I want to know how do they get the answer. I want to know how they get this…they got this. I don't understand how they got this.
>
> Von: You add it up; you do it and see what you get.
>
> Dana: I'm talking about this, and why did they do this?
>
> Von: I don't feel like sitting here…(inaudible)
>
> Dana: So I put…(*meaning the comment she writes on the Post-It note*).
>
> Von: You did an outstanding job…(*the comment Von suggests for the note*).
>
> Will: No, just this (*suggesting a comment for the following note*) we don't understand…(inaudible).

Dana continued in her quest to understand the nature of the difference, despite Von's encouragement. She remained focused on how the group calculated the area.

Dana: Look, they said length times the width here…they multiplied the length and the width.

Von: Could we just agree on something?

Will: No, just write: We don't understand how they got the maximum area. Write that.

Von: Write that? We don't understand how y'all got your maximum area.

Dana: Ok, yes I do. I got it. They all right. 'Cause they said they got 26, they multiplied 26, so they multiplied this one for, um…

Dana realized that Shay's method for calculating the area was correct. Her commitment to *truth* is evident in this episode, which we believe demonstrates her mathematical integrity. At this point, the teacher joined in on the conversation.

Teacher: So is this the same as yours or different from yours?

Dana: They are different.

Teacher: It's different.

Dana: Yeah, 'cause we multiplied…(inaudible)

Teacher: So, what do you guys have to say about what you see?

Dana: That they was right.

It is very important to note that, per the above dialogue, Dana agreed that Shay's group's solution was correct. This is further supported by data from an interview conducted several days later, wherein Dana noted that, at this point in time, she realized how to calculate area and knew that what Shay's group did was in fact correct. Analysis of the data (interview, classroom, field notes) lead us to believe that Dana's primary structure was "Check This Out", as she tried intently to understand Shay's group's solution,. It now appeared that for Dana, the mathematics became figure, and the social ground (she no longer needed to monitor her group's progress). Shay's primary structure remained "Check This Out", as he openly explored each group's solution.

Shortly after the above dialogue, Dana noticed that Shay's group was looking at *her* group's work. She listened to their comments and realized that they were saying that her group's work was wrong. In the excerpt that follows, the exchange between Dana and Shay became quite heated. We suggest that the "Don't Disrespect Me" structure was evoked for Dana, as her primary structure, when she heard Shay criticize her work. Their gestures and facial expressions were indicative of the feelings of fear, anger, and contempt associated with this structure. Dana was adamant in her response to Shay, and acted defensively regarding the accuracy of her solution.

Dana: (inaudible)…where it is wrong?

Shay: 'Cause you put, when you finding, um, the area, you timesed the width times the (inaudible).

Dana: All right, but we timesed all that up.

Shay: But you're not supposed to.

Dana: Alright, but we did it though, so…

Shay: But you're not supposed to, so it's wrong.

Dana: No, it's not wrong. Actually, no it's not.

Shay: It's wrong, it's wrong, **it's wrong**.

Dana: No, it's not.

Ghee: How is it wrong?

Shay: Look at that (*showing on calculator*), that's what y'all got, 16.

Dana: Yes.

Shay: That's what y'all got?

Dana: Yes.

Shay: So the width times the 40.

Dana: Well, we didn't do that, it equals 400, but we didn't do it.

Shay: 40, yes, that's how you do it. 40 times 10, that's how you get that.

Dana: Well, we didn't do it that, so, oh well.

Note that at this point, the dialogue revolved around the issue of would concede. Shay told Dana that she was supposed to multiply the width times 40. If Dana was interested in exploring the idea, as she might be if she were engaged in a "Check This Out" structure, she would have tried to understand why the multiplication suggested by Shay made sense. Instead, she said, "Well, we didn't do that." In other words, she was looking for a justification for her group's choice that would prevent any further humiliation rather than being engaged in a search to expand her understanding. Shay, in turn, who had been motivated initially by a desire to understand the problem had now become engaged in a new structure, "Look How Smart I Am." Shay said to Dana, "That's how you do it," indicating that what he wanted out of this interaction was a concession by Dana that his way of thinking about the problem was better than her way. His desire to extract a concession from Dana merely strengthened her resolve to avoid being humiliated. The "victim" of this interaction was mathematical understanding, which had been pushed to the background, as social jockeying became elevated to the foreground.

Shay: That's the right answer.

Ghee: 40 times 10?

Shay: Yes, that's the same thing they got, too.

Dana: So, for, you telling me that…whatever…

Ghee: 40 times 10?

Shay: 40 times 10 equals 400.

Dana: Oh well…

Dana: We already know 40 times 10 equals…

Shay: That's what your area is. Ok, that's what your area is.

Ghee: 40 times 10…you said 40 times 10? 40 times 10, 40 and 10…none of that add up to 100.

Lar: Yeah, so…

Shay: You add…you add this, that's 40 (*raising his voice*). That's 80 right there. And that's 100. Yeah, so you don't know what you're talking about. **That's the perimeter, that's the perimeter** (*almost shouting and moving closer to Ghee's face as he points to the work*). And area is 400.

Ghee: No, 80…(inaudible)…that's exactly what we did.

Shay: And that was wrong.

Dana: 80 times 20 is not wrong from 100 (*raising her voice and speaking with conviction to Shay*).

Ghee: 80 plus 20 is 100.

Dana: Thank you! **Thank you**! 80 times…I mean, 80 plus 10…I mean, 20, is 100.

Ghee: So now who got ripped? Shut up, man!

Dana: So wait…80…

Shay: **Exactly, tell 'em you gotta multiply length times width** (*shouting at Dana*). Not what y'all did.

Dana: **That's what we did! That's what we diiiid** (*shouting at Shay*)!

Ghee: That's exactly what we did! (inaudible) (*makes angry motion*)

Several days later, during the follow-up interview, Dana was asked to watch a video clip of the above interaction and reflect on it. The following transcription comes from that interview:

Interviewer: So what do you think?

Dana: That was a funny clip.

Interviewer: How were you feeling during that…?

Dana: Mad angry. Everything.

Interviewer: Well, why?

Dana: Because, Shay was proving me **wrong**.

Interviewer: So how did he prove you wrong?

Dana: He was **trying** to prove me wrong. Maybe he was right, and then maybe I wasn't right. Or maybe I was right, and then he wasn't right. I don't know. 'Cause he multiplied the length and the width, but I multiplied both of the lengths and both of the widths. So…

Interviewer: So what do you think?

Dana: That…maybe his answer is wrong. Maybe he multiplied the length and the width, because on the paper it said multiply the length, and not the width.

Note that at this time, Dana stated that Shay's answer *could* have been wrong, which is in direct contrast to her comments to the teacher during the actual class. However, in the next excerpt, she stated that it was possible that she was wrong. She also underscored the fact that she was "mad at him [Shay]".

Interviewer: So what do you think? What did you understand from that day?

Dana: That maybe I was wrong. I don't know whether I was wrong or right. I was just, that day. He was, I couldn't say nothing to him cause he…I was mad at him.

Interviewer: Were you feeling comfortable that day?

Dana: No.

Interviewer: Why?

Dana: Cause he was trying to prove **me** wrong.

Interviewer: Do you feel that way often? Does this happen often?

Dana: Yes.

Interviewer: So you're uncomfortable often?

Dana: Not all the time. But when I'm right, I'm not uncomfortable. But when I'm wrong, when they try to prove **me** wrong, I'm uncomfortable.

Interviewer: That's interesting. So when it's that, when are the moments when you get mad? When what's happening?

Dana: When they, when people in my group really aren't doing nothing at all, that makes me mad.

Interviewer: Is there anything else that makes you mad?

Dana: Uh…yeah, when people try to prove **me** wrong too.

During the interview Dana said that she was uncomfortable "…when people try to prove *me* wrong." Notice that she chose to use the word "me" rather than saying that she is uncomfortable when people try to prove her *idea* wrong. Shay's actions or words gave her the feeling of being challenged, thereby changing her primary focus from inquiry to saving face. Herein lies the heart of this structure. A student becomes interested in a mathematical idea but then encounters a situation where it is critical for him or her to save face.

Dana initially appeared to be interested in exploring a mathematical idea, seeking mathematical truth (integrity), and trying to understand how Shay's group solved the problem (mathematical intimacy). Her primary emotion changed to one that was fearful of looking foolish, or publicly losing face (a challenge to her self-identity). Shay's response to her group's work was the behavior that seemed to trigger the "Don't Disrespect Me" structure and derail her attempt to continue to productively explore the problem. Instead of meaningful exploration, an engagement structure designed to protect honor was elicited. As Shay pointed out the flaws in her group's solution, Dana seemed to lose face and her desired identity as an effective group leader (who was good at leading them toward a productive and complete solution), appeared to be damaged.

We believe that the above is a case where two of the affect components, self-identity and integrity, came into conflict. We suggest that Dana's investment in her identity as an effective leader of her group conflicted with her ability to elevate the mathematical integrity of the solution to its proper place. Investing in identity enhancement was done at the cost of sacrificing the integrity of the solution. Any solution that enhanced her identity as an effective leader was better than a solution with integrity that jeopardized her identity as an effective group leader (especially since she was now in the "public" position of being exposed as leading her group toward an incorrect solution). Rather than intense engagement (intimacy) functioning in the service of uncovering mathematical truth, it was invested instead in protecting a vulnerable self-identity, which would have been damaged because of a loss of face.

We would speculate that if Shay had not made his comments when he had, Dana, upon her realization of the error in her group's work, could have effectively preserved her identity *and* integrity, as she led her group to a major revision of the work. Once the error was made public, the "Don't Disrespect Me" structure was evoked. However, due to her fragile and incomplete understanding of the problem, she was simply not in a position to argue from mathematical strength, certainly not to the extent that Shay could. (Recall that Dana had never progressed past the *image making layer*, and unlike

Shay, had not reached a "*don't need" boundary* (where she was no longer tied to the action of drawing the rectangle).

CONCLUSIONS

The descriptions above reveal the several of the many different ways that students may react during a problem solving session. We illustrated how Dana was moving along a path of exploration in "checking out" the problem but reached a branch point triggered by a threat of humiliation from Shay's response to her (and her) group's work. She moved from a path of mathematical curiosity and exploration to one containing emotional feelings of anger, perhaps shame, and a desire to avoid losing face.

Sometimes, the choice of a particular branch point may be influenced by the individual's goals, which, in turn are influenced by their beliefs about themselves—in what Dweck (2000) calls their "self-theories." Consider the branch point in which Shay reached an impasse in trying to solve the problem. A student who holds a self-theory that he has a *fixed quantity* of intelligence sees his inability to solve the problem as evidence that he is not as smart as he thought he was. In contrast, a student holding a view that his *intelligence is malleable* and can increase as he learns to solve challenging tasks will welcome the impasse and believe that by working hard and trying to solve the problem he can become smarter. Such, we think, was the case with Shay. As he encountered impasse trying to solve the problem, he became energized. He asked his group mates to get him more pencils and paper because he was motivated to overcome this impasse. The student holding the fixed intelligence theory will adopt a performance goal in which she will engage in activities that make her look smart and avoid challenges in which she believes she may fail. Accordingly, this student is likely to experience the impasse as frustrating, feel negative emotional feelings, and choose the path that involves cessation of productive work on the task. We speculate that this may have been the case with Dana. Instead of responding as Shay did, she adopted a structure in which she tried to garner praise from the teacher for 'doing the job" that the teacher assigned rather than solving the problem. Such activity may involve formatting the page the way the teacher asked, including an answer to fill in each bullet point, and providing an explanation (even if incorrect) for an answer. A teacher who sets up the classroom environment to reward compliance with the task demands will encourage this pattern of responding to impasse. In contrast, the student whose theory is that intelligence is incremental, will relish the challenge and persist at the task despite her initial inability to solve the problem. Dana was succeeding in getting the job done and would have been pleased with this outcome had she not encountered Shay's response to her group's work.

The engagement structures we posited were theoretically interesting to us and helped us make sense of what we observed in the class and videos. At the same time, we felt uneasy about attributing inner states to actors without some external source of verification for our speculations. As a result, for the past year, we, along with Gerald Goldin, have been engaged in developing a questionnaire, which we have now administered to more than 500 students in over 25 classrooms (involving 27 teachers across six school districts—with students reflecting the geographic, demographic, and socio-economic diversity of New Jersey). They have all been working on a task that we have standardized so that we can observe variations in affect in conjunction with efforts to solve the same mathematical problem. The questionnaire we have developed is based upon an expanded theoretical articulation of different affective structures (listed above). For each, we have described an initiating event and a variety of stages in the evolution of an idealized pathway that unfolds over time. We have created questions that tap the various stages of the pathway for each structure. Additionally, we have listed emotional feelings that students can select to represent how they were feeling during key moments in the videos we observed. Finally, the questionnaire also allows students to identify salient thoughts they may have experienced during the session, which may have guided their actions. Preliminary findings are very encouraging. They help us make greater sense of the classroom interactions that we observe, and contribute to our greater confidence in drawing the sorts of conclusions that we have offered in this paper.

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