Lisa B. Warner

"How do students' behaviors relate to the growth of their mathematical ideas?"

HOW DO STUDENTS' BEHAVIORS RELATE TO THE GROWTH OF THEIR MATHEMATICAL IDEAS?

Lisa B. Warner

Rutgers University

The purpose of this study is to analyze the relationship between student behaviors and the growth of mathematical ideas (using the Pirie-Kieren model). This analysis was accomplished through a series of case studies, involving middle school students of varying ability levels, who were investigating a combinatorics problem in after-school problem-solving sessions. The results suggest that certain types of student behaviors appear to be associated with the growth of ideas and emerge in specific patterns. More specifically, as understanding grows, there is a general shift from behaviors such as students questioning each other, explaining and using their own and others' ideas toward behaviors involving the setting up of hypothetical situations, linking of representations and connecting of contexts. Recognizing that certain types of student behaviors tend to emerge in specific layers of the Pirie-Kieren model can be important in helping us to understand the development of mathematical ideas in children.

Introduction & Theoretical Framework

This research focuses on behaviors that relate to the growth of mathematical ideas in a group of middle school students, as they work on a specific task in an after-school setting. Students' behaviors are identified and then tied to an existing theoretical framework for the growth of mathematical understanding, in order to document the nature and progression of students' ideas (the Pirie-Kieren model, Pirie & Kieren, 1994). Analysis of students' movement through the layers of this model is used to identify how these behaviors relate to the overall process of understanding.

Pirie-Kieren Theory for the Growth of Mathematical Understanding

The Pirie-Kieren model for the growth of understanding (Pirie & Kieren, 1994) provides a framework for analyzing student growth in understanding, via a number of layers through which students move both forward and backward. Pirie (1988) discussed the idea of using categories in characterizing the growth of understanding, observing understanding as a dynamic process and not as a single or multi-valued acquisition, nor as a linear combination of knowledge categories. Pirie & Kieren (1994) illustrate eight potential layers or distinct modes within the growth of understanding of a specific idea by a specific person. They are as follows: primitive knowing, image-making, image-having, property-noticing, formalizing, observing, structuring and inventizing (see figure 1).

FIGURE 1 HERE

"Understanding" is a process that is not static, not linear, and often involves "folding back" to inner-layers, working there, then moving forward (see figure 2). Each time the learner "revisits" an inner-layer, he/she "revisits" it with the understanding acquired from the outer-layer (Pirie & Kieren, 1991).

FIGURE 2 HERE

The theory is a way to explain understanding and can be a useful tool for considering how "understanding" grows by providing a common 'vocabulary' to describe different layers of understanding. It is important to clarify that an idea may continue to grow over the course of many problem solving experiences and/or real life experiences, or as the person explores the idea in different contexts. In this paper, the term outer-layer refers to the last few outer-spirals in the diagram above (see figure 1), with the exception of structuring and inventizing (not addressed in this paper) and the term inner-layers is used to represent the first few inner-spirals in the diagram above (see figure 1), with the exception of primitive knowing, as will be explained below.

Pirie and Kieren describe primitive knowing as follows:

> The process of coming to understand starts at a level we call primitive knowing. Primitive here does not imply low-level mathematics, but is rather a starting place for the growth of any particular mathematical understanding. It is what the observer, the teacher or researcher assumes the person doing the understanding can do initially (Pirie & Kieren, 1994, p. 66).

Thus, they explain all understanding as having its roots in primitive knowing with the exception of what the person already knows about the specific idea. Any knowledge about the specific ideas is in one of the other layers.

How Student Behaviors Relate to the Pirie-Kieren Theory

The main research questions that are addressed in this paper are: Are different types of student behaviors associated with the growth of mathematical ideas in specific ways? If so, how?

To answer these questions, it was important to identify the student behaviors that appeared to be associated with the learning process. The student behaviors below were chosen because they emerge in the data as being associated with the growth of ideas and also, existing literature (see below) highlights these behaviors as important in the learning process (eg. Carpenter and Lehrer, 1999; Cobb, 2000; NCTM, 2000; Sfard, 2000; Kiczek, Maher & Speiser, 2001).

Explaining, Using, Reorganizing and/or Building on an Idea;

Prompting students to talk about mathematics is an important goal of education (NCTM 2000; Sfard, 2000; Doerfler, 2000; Cobb, Boufi, McClain, and Whitenack, 1997). Carpenter and Lehrer (1999) state that "the ability to communicate or articulate one's ideas is an important goal of education, and it also is a benchmark of understanding" (p. 22). Cobb (2000) notes that student exchanges with others can constitute a significant mechanism by which they modify their mathematical meanings.

Questioning an Idea and Showing That it is Valid or Invalid

As a student's ideas grow, he/she might explain his/her own or someone else's thinking, use it, reorganize it and/or build on it. He/she might also question it and/or try to show it as valid or invalid (justification). At times, argumentation plays a role in how these behaviors are enacted in the classroom. Whitenack, J.W. & Knipping, N. (2002) address the issue of the interdependent relationship between arguments and the possible learning opportunities that students might have as they engage in these types of discussions. The authors illustrate their ideas by providing an analysis of a student (Casey), using argumentation as their lens. When students did not understand Casey's explanation, it was necessary for Casey to justify his ideas. As Casey provided support for why his new ideas were relevant, he refined his thinking to more formal reasons for his solution.

Yackel (2002) shows how actions, diagrams and notation, as well as verbal statements, can function as a part of an argumentation. She highlights the teacher's role in recognizing the importance of warrants and backings and eliciting such argumentative supports. In 2002, Speiser discusses his claim that arguments relate to the development of understanding and analyzes how three students made their points (two students invited others to participate in how they reason and the third student reports on how she entered another student's way of working).

Researchers such as those cited above and others (see for example, Schorr, 2003; Maher, 2002; Shafer and Romberg, 1999; Warner & Schorr, 2004) also highlight meaningful student-to-student discourse. They maintain that it is important to provide students with opportunities to discuss their ideas with each other as they question and defend their own and each other's mathematical thinking and reflect upon the mathematical thinking of others. One important component of this involves students' questioning the mathematical thinking of their peers. Warner & Schorr (2004) found the following:

> When students have the opportunity and are encouraged to question each other about their mathematical ideas, both the questioner and the questioned have the opportunity to move beyond their initial or intermediate conceptualizations about the mathematical ideas involved. As students reflect on their own thinking in response to questions that are posed by their peers they have the opportunity to revise, refine, and extend their ways of thinking. As they do this, their earlier

conceptualizations and representations become increasingly refined and linked..(p. 429)

Using Multiple Representations and/or Linking Representations to Each Other

"The ways in which mathematical ideas are represented is fundamental to how people can understand and use those ideas." (NCTM, p.67) Representations may include pictures, tables, graphs, diagrams, symbols, etc. (NCTM, 2000). In the present study, the term representation refers "to the act of capturing a mathematical concept or relationship in some form and to the form itself" (p. 67, NCTM, 2000). Speiser & Walter (1997) add a very important dimension in their description of a representation as a presentation to oneself, as part of an ongoing thought; or perhaps to others, as part of an emerging discourse. The development and use of various representations, whether verbal or written, are important for many reasons. For example, Kaput (1999) notes that as students use their own representations to reason and argue, they begin to link their own representations to abstract representations in an effort to justify ideas and they develop a sense of ownership of the mathematical ideas.

Speiser, Walter & Maher (2003) provide an example of how several inscriptions (things that people write and draw) become important means for focusing explanations that are emerging. Their findings suggest that exploring and establishing connections played a key role in developing mathematical understanding by the ways the representations have been used. Warner, Schorr, Arias, Sanchez & Endaya (2006) identify a relationship between teachers encouraging the student behaviors cited above (eg. students linking their own representations to abstract representations in an effort to justify ideas, etc.) and students' changes in representations, abstractions and movement towards generalization.

Using the Same Idea in Different Contexts

Many authors (Kiczek, Maher & Speiser, 2001; Powell & Maher, 2003; Uptegrove & Maher, 2004) report on students connecting contexts. Uptegrove & Maher (2004) report on five students' explorations of structural relationships between problem situations. In particular, they describe the case of students recognizing isomorphisms in two problem situations. Powell & Maher (2003) analyze the discursive interactions of four students to understand why they build isomorphisms to resolve a combinatorial problem. They focus their analysis on the isomorphism the students build

> …by identifying their discursive propositions about dynamical links that establish one-to-one correspondences between, on the one hand, objects and relations or actions in one system and, on the other hand, objects and relations of another system in such a way that an action on objects of one system maps to an analogous action on the corresponding objects in the other system. (p. 26)

In both of these cases, the authors support the idea that students connecting contexts to each other contributes to the growth of mathematical understanding.

Raising Hypothetical or Actual Problem Situations Based on an Existing Problem: Creating "What if…?" Scenarios

Students might sensibly raise hypothetical problem situations based on an existing problem. These are essentially "what if" questions, and involve imagined scenarios, regarded as similar to an existing problem. For example, Warner, Alcock, Coppolo & Davis (2002), report on a grade 6 student. In relation to building towers four high, choosing from 2 colors, the student asked: "What if we have three colors to choose from, instead of two?" This question indicates that the student was not stuck in the constraints of the problem but imagined, what was to her, a similar problem in which she could utilize three, rather than just two, colors.

Walter (2004) analyzes the types of questions students ask as their mathematical understanding grows. She describes a speculative student question as a question asked by a student "…to suggest potential. 'What if?', 'what would happen if?', or 'why don't we try this?' questions that provide direction for further inquiry are of a speculative nature" (p. 40). In her findings, she describes an increase in the number of instances that students asked speculative questions as they moved to deeper stages of understanding.

The present study builds on the literature above by highlighting the interplay of the student behaviors described above and their impact on the development of mathematical ideas. The results suggest that certain types of student behaviors appear to be associated with the growth of ideas and emerge in specific patterns. More specifically, as understanding grows, there is a general shift from behaviors such as students questioning each other, explaining and using their own and others' ideas toward behaviors involving the setting up of hypothetical situations, linking of representations and connecting of contexts.

The findings indicate that while all three students are unique, there were many instances in which they all exhibited similar types of behaviors when moving through similar layers. For example, there was an increase in the number of instances in which all three students linked representations to each other as they moved to outer-layers of understanding (compared to inner-layers).

Task Selection

Stein, Smith, Henningsen, and Silver (2000) suggest that certain types of mathematics problems are both *challenging* and *accessible* to students. Such problems have a high level of cognitive demand. By cognitive demands, they mean, "the kind and level of thinking required of students in order to successfully engage with and solve the task (p. 11)." They argue that it is the level and kind of thinking in which students engage that determines what they will learn. "Tasks that require students to perform a memorized procedure in a routine manner lead to one type of opportunity for student thinking; tasks that demand engagement with concepts and that stimulate students to make purposeful connections to meaning or relevant mathematical ideas lead to a different set of opportunities for student thinking. (p. 11)." A high cognitive demand task would seem to have the potential to elicit the student behaviors described above.

Classroom Environment

When studying understanding, the environment plays an important role. The National Council of Teachers of Mathematics (2000) supports problem-solving based classrooms, and the National Research Council (1989) suggests that students should be engaged in higher-level problem-solving activities that involve discussion and community consent for solutions to these problems. Many researchers describe environments where students are encouraged to discuss disagreements and differences, question and justify solutions, revisit ideas over time, generalize and extend ideas (cf. Maher, C.A. 1998 a & b,; Sfard, 2000; Dorfler, 2000; Cobb, Boufi, McClain, and Whiteneck, 1997, Schorr, 2003; Maher, 2002; Shafer and Romberg, 1999).

Methods

Background: The research reported in this paper is part of a larger study that investigated flexible mathematical thinking and multiple representations of students in an after-school mathematics program in New York City. In the larger study, ten sixth and seventh grade students, of mixed abilities, volunteered to participate for twelve weeks in an after-school mathematics program, in which students worked on high cognitive demand tasks (such as, “Building Towers” - *finding the number of possible 4 tall* towers *of building blocks, choosing from 2 colors of block*; Martino & Maher, 1999), which comes from the realm of discrete mathematics. It is important to note that all of the tasks explored by the students in the after-school sessions came from the realm of discrete mathematics and a few had the same or similar underlying structure as Building Towers, while others did not. All of these students were also taught by the teacher/researcher during the school day, where the students investigated some tasks that came from the realm of discrete mathematics, which included a few tasks with the same or similar underlying structure. They also explored tasks coming from different realms of mathematics with totally different underlying structures during the school year.

The after-school class was announced in each of the 5 math classes taught by the teacher/researcher. The parents of these 10 students agreed to have their children attend the twelve-week after-school math class. During these after-school sessions, the teacher/researcher encouraged the students to discuss disagreements and differences, question and justify solutions, revisit ideas over time, generalize and extend ideas (for more details, see Warner, Alcock, Davis & Coppolo, 2002). One month later (after the twelve-week sessions ended), individual and group sessions were conducted with half of these students (chosen by availability) after school and during their free lunch period. Two student-posed tasks were developed and investigated by the group during these individual and group sessions. The tasks that the students posed had the same underlying structure as the task the students investigated during the first few sessions of the after-school class (Building Towers-described above) and a similar underlying structure to another task (called Equivalent Trains-described below), which was introduced to the students towards the end of the twelve-week sessions: *Find all trains equivalent in length to each of the ten different colored Cuisenaire rods. For this activity, you can use any colored rod. Also, trains that are made up of the same rods, but in a different order, are regarded as different. For each color, convince us that you have them all.*

Many of the students found the generalized solution to "Building Towers" while attending the after-school math classes, which may have contributed to their ability to construct and pose math problems with the same underlying structure, months later. In the results section below, some of the students also refer to aspects of one of the student-posed tasks called "The Flag problem", which was the first problem generated by the students during the group sessions that took place after the twelve-week sessions ended (for more details, see Warner, 2005). This problem asks for *all possible flags with two stripes, choosing from 5 colors.* Another student posed an extension to this task by providing an additional *choice of 5 shapes for the corner of the flag and a choice of 5 pictures for the middle of the flag*.

In the present study, the ideas of three students were selected for deeper analysis- a sixth grade boy, David, who was enrolled in honors classes (in which students scored above the ninetieth percentile on the standardized test) during the school day, a sixth grade girl, Michelina (also in all honors classes) and a sixth-grade girl, Anna, who is in all tier 2 classes (in which students scored below the fortieth percentile on the standardized test). Students at this middle school are placed in one of three tracks, as a result of teacher recommendation and standardized test scores. These students weren't selected for analysis based on these tracks (although both extremes were represented). These three students were selected for deeper analysis because they were available to continue to attend the group sessions for months after the twelve-week program ended (for a longer amount of time than some of the other students who continued). The extra exploration time could clearly be an important factor in the analysis and conclusions that follow. Having additional time to work on extensions of the task (similar in structure to the other student-posed task) seemed to afford them the time to move further out in the layers of the Pirie/Kieren model and therefore allowed for a more in-depth analysis of the impact of the student behaviors at these outer-layers, as well as the inner-layers. The present study focuses on the last three small group sessions that these students attended, as they investigated the Castle problem. "The Castle Problem" (student-posed) is stated as follows: *If you're at the beach and there are 3 different kinds of buckets in front of you* *(small, medium, large). How many different ways can you build castles if there are a total of 10 castles in front of you [parts of a castle]?*

Data: The data includes two camera views of each of these three sessions, descriptive field notes written immediately after each session and all associated student work.

Analysis: Powell, Francisco & Maher (2003) list a sequence of six interacting, non-linear phases of an analytical model for studying the development of mathematical thinking in which the analysis was modeled. These phases include describing the video content, identifying critical events, transcribing, coding, constructing a storyline and constructing a narrative. Descriptive summaries and transcription were written for both views of all three sessions. Since there were two different views of each of the three sessions that were chosen for deeper analysis, the different views were summarized, transcribed and analyzed by two different researchers, addressing issues of validity and reliability (LeCompte & Goetz, 1982). VPrism software was used as a tool to summarize, transcribe and code the data (Powell, Francisco & Maher, 2003).

Events that involved movement to a new layer of understanding or a change in representation, discourse or the raising of a hypothetical situation within a layer were selected as "critical events" for deeper analysis. A critical event acts as an organizer for the development of an idea, in that one looks at what took place just before and just after a designated critical event to construct a context for development of the idea. From these events, codes for student behaviors were developed (see table 1 below). Each critical event, with additional details that were needed to understand that event was identified as an "episode" in the present study.

In the present study, both camera views of the last three small-group sessions were summarized, transcribed & coded for student behaviors (see table 1) and the Pirie/Kieren layers (see table 2).

INSERT TABLE 1 HERE

INSERT TABLE 2 HERE

After the transcripts were coded using all of the codes (see table 1 & 2) above, connections between the layers and student behaviors became apparent. Miles & Huberman (1994) suggest that displaying data in different ways can help researchers uncover overlaps and commonalities. Consequently, several tables were constructed, which allowed overlaps and commonalities to be more readily revealed.

The episodes described below are selected as examples of all of the identified episodes (for full case studies and all episodes see Warner, 2005), which indicates any move to a different layer of understanding and/or change in representation, discourse and/or the raising of a hypothetical situation during this investigation of the Castle Problem. These particular episodes were selected for two reasons. First, some episodes were chosen because they are particularly good examples of the ways in which the student behaviors (described above) are associated with the growth of mathematical ideas, using the Pirie-Kieren model. Second, some episodes were included as examples to provide a storyline for the reader.

Results

The first section will focus on illustrating movement through some of the layers of the Pirie-Kieren model and the associated student behaviors, using episodes from Anna and David's case studies. Both of the students began with the same representations as they worked in the image making and image having layers and both moved from less sophisticated representations to abstraction and then to generalization, as they moved through the layers. There were a few differences, though. As Anna moved to the property-noticing layer, she often explained her thinking mostly in response to questions asked by the teacher/researcher. Her move to the formalizing layer and observing layer occurred with less explanation and more writing. David began in the image-making layer with less explanation and more writing but as he moved to the formalizing and observing layer, he articulated his ideas as he wrote, with fewer questions from the teacher/researcher.

The second section will focus on the ways in which Michelina moved through the layers, which was quite different from the other two students. Michelina's case is a good example of a student working in the image-making layer by trying to use a formula (representing it with symbolic notation) without any justification for its use. Michelina used a formula that David created for a problem they solved months earlier (Building Towers). As the teacher/researcher encouraged Michelina to make sense of her symbolic representation and justify her solution, she changed her representations and ways to use her symbolic notation several times as she made sense of her symbolic representation and moved to the formalizing layer. It is important to note that Michelina attended two out of the three sessions (the second and third session), while David and Anna attended all three sessions of the Castle Problem investigation that is analyzed below.

Anna and David's movement to the property noticing layer

Anna and David: Background

This section begins with episodes from Anna's case study. Figures 3a, b & c (below) illustrate some of the representations constructed by Anna as she explored the Flag Problem, a month prior to the episodes below. This background information is important because she and David both retrieve these representations as they investigate the Castle Problem in the episodes below.

INSERT FIGURE 3A HERE INSERT FIGURE 3B HERE

INSERT FIGURE 3C HERE

Below are some of the representations constructed by David as he worked in the outer layers, during his investigation of the flag problem (see figure 3d, e & f below). It is important to note that these representations are not the only representations David and Anna moved to as they solved the flag problem. For example, David began by drawing pictures of flags, using different colors for the stripes of the flag. His representations below are important, though, because Anna and David both use and/or build on these as they move to the formalizing layer, while investigating the Castle Problem below. David also uses his representations (figure 3g is an example) that he constructed as he solved Equivalent Trains (using cuisenaire rods) months earlier.

In the first few episodes below, the reader will notice that Anna and David both begin with similar representations (which were constructed by Anna) as they work in the image making layer of the Pirie-Kieren model. Both David and Anna retrieve and build on David's previous representations when they move to the formalizing layer.

INSERT FIGURE 3D HERE INSERT FIGURE 3E HERE

INSERT FIGURE 3F HERE

INSERT FIGURE 3G HERE

Moving to the image-making layer

Anna: Beginning of Session 1.

In this first section, Anna uses a previous representation (see figure 3a above) in a new context, which is associated with her move to the *image-making* layer. When the learner is "doing something" to get the idea of what the concept is, he/she is working in the *image-making* layer. Pirie & Kieren (1994) explain that a person must create an image before he/she reaches any other layer (except primitive knowing). A person working in the image-making layer is "tied to the action" or "tied to the doing".

David begins by reading a student's written version of the Castle problem.

David — If you're at the beach, and there's three different kinds of buckets, small, medium and large, in front of you, how many different ways there are [sic] [*David laughs*] to make them [castles] if there are only a total of ten buckets [later referred to by the students as parts of a castle].

Anna — So, they're saying that there's only ten of them.

T/R[1] — So you have ten castles in total and you have a choice of a small, medium or large bucket.

Anna — This is hard.

Anna immediately begins writing, and although she says that this is a difficult problem, she immediately begins with a useful representation and starts to solve (See figure 4a).

Anna — Small you can have…it's almost…small, medium…and large. [*drawing a tree with three branches coming out of each of the three sizes*]

Anna has already moved to outer-layers of understanding for this idea in a different context (flags), prior to this episode (see figures 3b & c). When Anna uses the words "it's almost", it appears that she is *folding back* (Pirie & Kieren, 1994) to *image-making* to retrieve a representation she used in a previous context. This facilitates her *image-making* in this context, using her previous representation (figure 3a), branching out the names of colors, to create an image for the use of the idea in this new context (figure 4a), using the names of bucket sizes (using letters to represent them) in the same manner.

INSERT FIGURE 4A HERE

David: Beginning of Session 1.

David uses Anna's representation (see figure 4a above) in a new context, which is associated with his move to the *image-making* layer. David begins by saying that he doesn't understand what the problem means. After Anna constructs the picture in figure 4a, David listens to Anna respond to questions about her representation (asked by the teacher/researcher) and answers some questions about Anna's original tree representation himself, in which she uses the words "small", "medium" and "large". As David begins to answer questions about Anna's representation, he begins to draw his own tree (see fgure 4e), which is exactly the same as Anna's tree (see figure 4a).

INSERT FIGURE 4E HERE

David's move to the image-making layer is associated with trying to understand another student's representation and then using it himself.

Moving from image-making to image-having

In this section, episodes from session 1 of Anna's case study and David's case study are selected to illustrate Anna's and David's moves to image having. It is important to realize that there may have been other moves to new layers, moments of folding back to inner-layers and/or additional student behaviors that occurred between these episodes (see Warner, 2005, for all episodes). This is also the case with the episodes chosen in the other sections that follow.

---

[1] T/R represents Teacher/Researcher

Anna and David: Session 1.

In the episodes below, Anna is questioned by David into reorganizing her ideas and using a previous representation in another context. Being questioned by David is associated with the reorganization of her ideas, move to a new representation, and move from *image-making* to *image-having*. "At the level of image-having a person can use a mental construct about a topic without having to do the particular activities which brought it about" (Pirie & Kieren, 1994, p. 66). One has an image and has reached a "don't need" boundary where he/she is no longer "tied to the action or doing". Anna continues the next part of her solution (figure 4b), which was also a representation she used for the flag problem a month prior (see figure 3b) used in a new context. Anna reorganizes her idea to use a previous representation in a new context.

INSERT FIGURE 4B HERE

In the event below, David questions Anna's answer of nine possibilities and her second representation (figure 4b). He offers a suggestion and they continue drawing their trees using his idea.

David Can I ask you a question? How come you did small small, small small and medium? …because then there's three castles. [*David points to her trees*.]

Anna [*connecting the small small, medium medium and large large tree and writing a nine at the end of it]*...but, there are nine possibilities.

David No, there's more than nine. Hold on let me…this would be…let me get another sheet here [*taking a new sheet*].

At this point Anna becomes excited - it appears that she recognizes her mistake. David begins to show her how to reorganize her idea by drawing additional possibilities.

David You would have small small. Then you would have small medium and small large.

Small, small and that branches three ways. [*He draws three branches coming out of small, small.*]

Anna Yes.

David So it's small small small. [*writing the word small coming out of a branch and drawing all possibilities for Small small, Small medium and Small large…, see figure 4f*]

Anna Medium…Now, would you do Medium, small?

INSERT FIGURE 4F HERE

David questions Anna's idea, points out what she missed and offers a way to revise it. Anna uses David's suggestion and reorganizes her tree (figure 4c) with what appears to be a deeper understanding of the relationship between her own representation and the context. David and Anna continue to draw their trees, separately, working in the *image-making* layer.

As David questioned Anna's use of her representation (as described above), he used Anna's representation (figure 4b) to move to a new representation (figure 4f). He built on her idea by color-coding his possibilities to present his organization in a more structured manner. Using another student's representation in this context is associated with a move to the image making layer for David. He is also tied to the action of writing small, medium and large as he creates his image.

Immediately after David questions Anna again, she corrects her mistake, and then uses a previous idea (Figure 3c) to move to a new representation (Figure 4d).

David How come you wrote small small, medium medium, medium small, medium small?

Anna What do you mean? Small small, small medium, small large, medium small, medium small, oh. [*Anna fixes the second medium small and writes medium medium, instead*.]

INSERT FIGURE 4D HERE

After Anna corrects this mistake, she changes her representation to a "three tree" (Figure 4d), which is associated with her move from *image-making* to *image-having*. She no longer needs to write small, medium and large; she has an image of the three sizes and writes a "3" to represent them. Being questioned by David coincided with Anna's understanding of her own mistakes, which contributes to a move to a new layer of understanding (*image-having*).

In the episode above, David questioned Anna's representation, then explained why it was valid. After questioning Anna's new representation, he explained this representation himself. As Anna moved to a new representation, he builds on her idea (see figure 4g).

David: Oh, I think I know what you're doing. You started off with three kinds [*pointing to her first three*] then you did that [*following the three branches down with his finger*], then there's three more.

Anna: Yeah.

David: That's six though. [*He places his two fingers on both of the 3's branching out 3 times each. Anna draws another branch coming from the original tree and writes a three at the end of it*.]

David: Yeah.

David: [*inaudible*] That's like a simplified Pascals.

David writes a nine with nine branches coming out of it (see top of figure 4g).

INSERT FIGURE 4G

Anna questions David's idea and shows it is invalid by providing a reason for using the three in terms of the properties noticed about her images.

Anna Wait, no, first you should do three.

David — Wait, let me explain. Nine, sort of like, this would be…are you saying...maybe nine is not the right number?

Anna — Yeah, I think three.

David — Right. I'm sorry it is three [*draws a line under his 9 with 9 branches*].

T/R — Why three?

Anna — Because there's three, there's small, medium and large.

David — Right.

Anna — So, out of that three you get three more and it's…you get three from small, three from medium and three from large [*points to the 3's in the 2nd layer of her tree*]. Then from that you get three from small, three from medium, three from large [*points to the 3's in the third layer of her tree*].

David reorganizes his representation after his use of a nine was questioned (see figure 4g). His new representation uses numbers, instead of words to represent the bucket sizes. He builds on Anna's representation and retrieves his own previous representation in which he used Pascal's triangle to solve the building towers problem (see figure 3g above). He is now working in the image having layer because he now has an image of the bucket sizes and is no longer tied to writing out the words.

Moving from image-having to the property-noticing layer

David: Session 2

In this section, the following episode is selected to illustrate David's move to property noticing. Pirie & Kieren (1994) describe the property-noticing layer as, "When one can manipulate or combine aspects of one's images to construct context-specific, relevant properties" (p. 66). "This involves noting distinctions, combinations or connections between images, predicting how they might be achieved and recording such relationships" (Pirie & Kieren, 1992b, p. 247). This would be looking at the images and saying, "How are they connected?"

In this episode, David explains the relationship between the numbers in his "Pascal's triangle with 3 sizes" representation and what each number represents, in terms of individual castles. He described what he means by a three-way fork, and the way the castles from previous possibilities fall on top of other castles in each direction when forming new possibilities when one more part of a castle is offered. He compares this to the way the rods fell in "Equivalent Trains". David's move to the property noticing layer occurs in association with his use of previous representations in a new context (exponents & the way he uses Pascal's triangle to find all possibilities), linking representations to each other (letters, Pascal's triangle and exponents) and linking contexts to each other (castles to equivalent trains).

David: This is just like using the Pascal's tree to show the buckets, not the buckets, the castle order. So, I wrote small medium large, then I went to ss, sm and sl, which was small small, small medium, small large.

David: This is a chart yeah, except it's going, it's a three way fork. [*See figure 4h*]

INSERT FIGURE 4H HERE

T/r: Go ahead, it's a three-way fork. Can you explain what happens at the fork?

David: Well, it either falls this way, this way or this way [*tracing the right, middle and left branch with his marker*].

T/r: And what happens when it falls to the right?

David: Well, if it falls to the right...it goes onto the large, and when it falls to the center it falls on a medium and when it falls to the left, it goes on a small.

Michelina: See you're smart, all I do is list them.

David: I, umm, remember, I did this when it was in the rods, I color-coded the way it fell, when it fell this way, orange would make small, green would make medium and blue would make large.

David explains how his representation is used for castles and the described the properties (e.g. the possibilities can be generated systematically by using the idea of a three way fork). He describes how the previous castles fall on top of a new castle to form the number of possibilities in the next position for the next row. He also began to connect this to his previous use of this representation for the context of trains.

David's work shows that he used numbers to represent the sum of two adjacent numbers in the previous row, which represented one less component offered. When the possibilities in the previous row "falls" to the left, a small is added to the bottom. When they fall to the middle a medium is added to the bottom. When they fall to the right, a large is added to the bottom of the previous possibilities. Then, he added up all possibilities that join to get the number that he wrote. He was now using this image to keep track of the possibilities in an abbreviated way (grouping them together), rather than having to think about these castles individually again

Anna: Session 2

In this section, the episode below is selected to illustrate Anna's move to property noticing. Here, Anna's move to the property-noticing layer occurs in association with her moves to new representations, setting up of hypothetical situations and connecting of contexts. Anna connects the sizes in the Castle problem to the colors and shapes in the Flag problem by re-explaining her hypothetical situation she created for Flags. She then builds on this idea by creating a similar hypothetical situation for Castles. When Anna describes exactly what the tree would look like without drawing it, it appears that she has an image of what her tree representation would look like for castles, choosing from 4 sizes. She builds on her old hypothetical situation by adding another size (XXL), choosing from five sizes, and notices even more properties about the images she has.

Anna	David's problem [referring to the flag problem] would be the same thing except we're talking about size and he's talking about shapes.
T/R	Hmm, so how is David's problem the same?
Anna	Ok, like you see, I moved on and you can go on with this problem too…like in David's problem I said what if you add another color or add another shape? …In this one, you can say what if you add another size.
T/R	Maybe you can work on that, too, what if you did add another size?
Anna	It would be four [*tapping her pen next to the three branch*], then four comes from that, [*drawing the four branches with the pen in the air*], that would be four, eight, twelve, sixteen [*tapping her pen on each imaginary spot that the four branches would be in on her paper*]…and then out of those four another four come out. It's the same thing [*finding her previous work for flags, figure 3c, and pointing to it*]. I say five over here [*pointing to the top five*] and then five come out [*pointing to the five branches with her pen*] and for each five you add five more.
T/R	Hmmm. So what would that five represent in Castles?
Anna	You would need…small, medium, large, extra large, extra extra large.

Anna sets up new hypothetical situations for the existing problem (what if there are 5 sizes to choose from), refers to a previous hypothetical situation she set up for another context (what if there are 4 colors or shapes to choose from), and connects these representations to each other. She was working in the *property-noticing* layer, as she connected these images (her description of what the tree would look like for a choice of 4 sizes and how a 5 tree would look for the context of the castle problem) to each other.

Michelina's movement to the formalizing layer

Michelina: Background.

Below is Michelina's student work from the building towers task (figure 5a) and the flag problem (figures 5b, c, d & e). These are important for two reasons. First, she retrieves and/or reorganizes some of these representations in the episodes below. Second, it is important to see that the confusion she has with her use of X^y in the episodes below was also apparent as she struggled through using the same symbolic representation when she solved the flag problem (see figures 5b, c, d & e).

INSERT FIGURE 5A HERE

INSERT FIGURE 5B HERE

INSERT FIGURE 5C HERE

INSERT FIGURE 5D HERE

INSERT FIGURE 5E HERE

Michelina's movement to the formalizing layer

Michelina's movement to the formalizing layer is very different than in the case of the other two students. In Michelina's case, she began in the formalizing layer and repeatedly folded back because her knowledge didn't appear to be stable in the formalizing layer. Therefore, the main aspects of the most prominent layers are identified in the selected episodes below.

Michelina: Session 2.

As soon as Michelina was introduced to the castle problem, she immediately tried to retrieve a formula (see figure 6a), previously constructed for the context of towers and flags, expressing her thoughts symbolically as X^y.

INSERT FIGURE 6A HERE

Michelina expresses her conjecture symbolically and initially seems confident in her use of her formula (see figure 6b) by solving a hypothetical situation she set up by using her formula. Therefore, she appears to be working in the formalizing layer.

INSERT FIGURE 6B

When asked to justify this, she becomes very confused about how the base and exponent relates to this context and is not sure about how to justify this conjecture. That was when the teacher/researcher asks her to pose another hypothetical situation that she might try to justify. Michelina says that if she was asked to choose from five bucket sizes and had three parts to her castle, there would be a total of 125 possibilities. When Michelina was asked to provide a convincing argument, she tries to retrieve her formula and becomes confused about which number should be used as the base and which number should be used as the exponent (5^3 or 3^5) to find all possible castles- (see figure 6c & d).

INSERT FIGURE 6C INSERT FIGURE 6D

It can be inferred that Michelina folds back to the image-making layer because her knowledge was not solid in the formalizing layer. Michelina's move to the image-making layer is associated with using her own previous representations and ideas of others and setting up hypothetical situations based on the castle problem.

Michelina: Beginning of Session 3.

In this episode, Michelina tries to justify her use of the formula $N=X^y$ by attempting to link tree and exponential representations to each other. In her effort to link representations to each other, she changed her use of $N=X^y$ in this situation, interchanging the numbers used as base and exponent (3^5). In the following transcript, her symbolic representation was questioned first by others, then by herself.

T/R You branch out a small, medium, and large. So why is it then that you don't use the extra small and the extra large with the extra small?

David: You do

Michelina: No, you don't

David: Yes you do

Michelina: No you don't

David: The exponent, that's not what the exponent means.

Michelina: Look, because I'm always branching three times, not five. I only have three, not five.

David: Why, why don't you… why don't you branch out five?

INSERT FIGURE 6E HERE

Michelina is working in the image-making layer, trying to construct an image that explains either 5^3 or 3^5. After this interaction, Michelina adds another 2 sets of trees (XL & XS, with 5 branches coming out of each) and then crosses them out (see figure 6e above). After crossing them out, David tells her she is incorrect and writes “Undo”. After David and Michelina discuss this idea some more, Michelina went back to her initial use of $N=X^y$ in this situation, using the base as the choices of buckets and the exponent as the number of parts of a castle (5^3), with a little more certainty.

Michelina's movement to the formalizing layer

Michelina: End of Session 3.

When formalizing, "the person abstracts a method or common quality from the previous image-dependent know how which characterizes his/her noticed properties" (Pirie & Kieren, 1994, p.67). This is when he/she creates a "for all" statement or "formal statement". When one is formalizing, one no longer needs to talk specific but can make a general statement.

In the episode below, as Michelina correctly links a tree representation to her formula ($N=X^{y)}$ for finding N castles with X options and y components and moves to new representations (see figure 7a & b), she moves to the formalizing layer. David's questions (in the episode above) and her own questions are associated with Michelina's move to new representations and linking of representations to each other, which all relate to her justification of a correct formal statement (see figure 7a & b).

INSERT FIGURE 7A & B HERE

Michelina links her use of exponents, multiplication by five, tree representation and numbers used to represent each level, as she justifies the use of $N=X^y$ in this situation. She also explains the relationship between these representations (see figure 7). She writes, "For each of the 5 sizes, I will branch off another 5 times to represent the 2^{nd} level or 5^2. Then, after that, I will branch off each of those 5 into another 5, which makes my exponent change from 5^2 to 5^3." This indicates her move to the formalizing layer in which she abstracts a method or common quality from the previous image-dependent know how which characterizes her noticed properties (how each layer of her tree relates to her multiplication number sentence and symbolic notation).

Moving to the observing layer

This section highlights the relationship between Anna's and David's behaviors and moves to *observing* - noticing properties about "for all" statements. This layer involves "…observing one's formalizing and organizing these observations" (Pirie & Kieren, 1992a, p. 247). This parallels property-noticing but at a higher level. "A person who is formalizing is also in a position to reflect on and coordinate such formal activity and express such coordinations as theorems. We call such an understanding observing" (Pirie & Kieren, 1994, p.67). When observing, one looks at the "formal" statements and asks "How are they connected?"

Anna: End of Session 3.

Prior to this episode, Anna recorded a statement, 5^n, and the statement was a formal symbolic way of describing the situation as 5^n ways of building castles when there are 5 choices of bucket sizes. She writes 5^n and makes sense of her symbolic notation by finding the total number of castles for n number of components when there are 5 sizes to choose from. She also writes the expression X^y (labeling X as the number of bucket sizes & y as the number of castles), and makes sense of this symbolic notation by finding the total number of castles for y components, when there are X options.

Anna sets up a hypothetical situation & connects contexts in figure 8a & b below by investigating what would happen if there were 4 sizes to choose from and 5 sizes to choose from. In figures 8a & b, Anna implies that if she adds another choice of size (for castles) or choice of color (for towers or stripes on flags) to 3^n, she will get 4^n. She continues by showing how her representation of 4^n relates to 5^n. She links representations to each other by using powers of 4 and 5 (using exponents) to label each layer of her tree & explains in writing how multiplication relates (eg. "In each 4 there is 16"), showing the relationship between multiplying by the base, her tree representation and the exponent. All of these representations & labels can be seen in Figure 8a & b, which were written by Anna at the end of session three.

INSERT FIGURE 8A&B HERE

Anna’s move to the observing layer is associated with setting up hypothetical situations for the existing problem (for example, “If I add another size or color to 3 to the n, I get 4 to the n”), by connecting representations (for example, using exponents to label each layer of the tree) and connecting contexts (for example, labeling each “n” and base with the information that it represents in Castles, Towers and Flags). As she sets up hypothetical situations, connects contexts and links representations to each other, she is noticing the connection between her formal statements and properties they have in common.

David: End of Session 3.

In this section, David explains his symbolic notation for finding all possibilities in several different contexts (figure 9). He begins by describing the underlying meaning of the base in the context of equivalent trains (see description of task in Methods section).

INSERT FIGURE 9 HERE

David: Two, when we use 2 as a base, we show how numbers can double. So, when we show three as the base, it's tripling; four it's quadrupling; five it's quintupling, and so on and so forth.

David sets up a hypothetical situation (“What if the base were 3, 4 or 5?”) and begins by describing what the base means in each of these situations. Then, he continues by linking this representation to his formal statement he created, written in symbolic notation, for finding all possible equivalent trains.

David: …so, the X to the y in this, the X was used to show doubling.

INSERT FIGURE 10 HERE

David: No, it was always 2, though [referring the trains task]. It showed doubling in this one. But, in like, uhh, the flag problem it was 5 because it was showing that there were five different changes..." [*David continued by explaining what the base represents in terms of flags, connecting contexts*.]

INSERT FIGURE 11 HERE

___David explains that in the context of equivalent trains the base would always be 2 and each color would always branch out 2 times but for the other contexts, the base would change and always indicate the number of changes and/or branches (See figures 12a, b & c).

INSERT FIGURE 12A INSERT FIGURE 12B INSERT FIGURE 12C

As David links representations to each other (eg. tree diagram, number sentence, symbolic notation, his use of Pascal's triangle to find all possibilities, etc.), set up hypothetical situations (eg. "What would it mean if the base is 5?") and connects contexts (eg. "How would I use X^y to find the total number of possibilities in each context?"), he noticed properties about his formal statements and how they relate to each other for each context. I infer that he begins to notice the similarities in the underlying structure of each of these contexts. David is on his way to the structuring layer, in which one would explain one's formal observations in terms of a logical structure (Pirie & Kieren, 1992a). Pirie & Kieren (1994) state, "Structuring occurs when one attempts to think about one's formal observations as a theory. This means that the person is aware of how a collection of theorems is inter-related and calls for justification or verification of statements through logical or meta-mathematical argument" (p.67).

Summary of Findings

The relationship between student behaviors and folding back

The three students all re-explain previous ideas and re-organize or build on previous ideas as they fold back to inner-layers. Anna also connects contexts as she folds back to inner-layers of understanding. Both Anna and David also often move to new representations while folding back to inner layers. An example of this is provided below.

Folding back to image-making from formalizing

David: Session 3.

Some of these student behaviors are also associated with the action of "folding back" to an inner-layer. For example, as David works in the formalizing layer and attempts to apply X^y to different contexts, he folds back to the image-making layer, and continued to work there for a while, to make sense of how X^y related to a new context. In attempting to figure out how to label his variables, David draws a picture of one possibility (see figure 13), using a previous picture representation he constructed, as he folds back to the image-making layer. It is important to note that he folds back with all of the knowledge he developed in the formalizing layer and this fold back contributed to his move back to the observing layer in later episodes.

INSERT FIGURE 13 HERE

David's fold back to the image-making layer is associated with a move to a picture representation. As he links this picture representation to his symbolic representation (showing what one possibility looks like, to be sure he is using the base and exponent in the correct way) in an effort to make sense of his formal statement, he quickly moves through the image-having and property-noticing layer, back to the formalizing layer (with this new knowledge).

The relationship between student behaviors and the layers of understanding

To examine this relationship more clearly tables were generated, which display the behaviors that were associated with moves to each layer and/or working within a layer, for the three students. Tables 3 and 4 (below) display all observed student behaviors (for all observed episodes, see Warner, 2005) that occurred as the three students moved to a specific layer or worked in a specific layer during the 3 day problem solving experience, solving the castle problem.

A summary of the observed behaviors and the associated layers in the Pirie-Kieren model (based upon the detailed case studies of the development of the three students' thinking[2]) follows[3]:

INSERT TABLE 3 HERE

INSTER TABLE 4 HERE

- There was a decrease in students re-explaining, questioning and/or using their own or others' ideas as students moved to outer-layers of understanding, with this being the most frequent behavior occurring when the students were working in the inner-layers of understanding.

[2] The totals and percentages in tables 3 and 4 represent all episodes described in Warner (2005).
[3] All percentages in table 4 are rounded to the nearest whole. Therefore, each column does not add up to exactly one hundred percent.

- Students moved to new representations fairly consistently throughout the problem solving sessions, with more instances of this occurring in the inner-layers of understanding.
- Students reorganized and/or built on their own and/or others' ideas fairly consistently as students moved to the property-noticing layer, with more instances of this occurring in the inner-layers of understanding.
- There was a marked increase in students linking representations to each other, with a low frequency in the inner-layers of understanding.
- There was a marked increase in students setting up hypothetical situation and connecting contexts, with very few instances of these behaviors in the inner-layers.

Conclusion

The findings suggest that, for these students, certain types of behaviors appeared to be associated with the growth of mathematical ideas in certain ways. For example, student questioning, explaining, re-explaining and using their own or others' ideas occurred with a higher frequency in the inner-layers of understanding and with a lower frequency in the outer-layers of understanding. During the inner-layers of understanding, one might expect students to question each other, because as students build representations that others do not understand, repeated questioning and requests for explanations and clarifications would be likely to occur more often.

Setting up hypothetical situations, connecting contexts and linking representations to each other occurred with a low frequency in the inner-layers of understanding and with a high frequency in the outer-layers of understanding. Again, this is to be expected as a student would be less likely to set up hypothetical situations when he/she is still trying to make sense of the problem. Similarly, a student would seem less likely to link representations until he/she has a solid and usable representation to link something to. It also follows that the students began to see the connection between contexts as they began to see the connection between the images they had (noticing properties) within this context, as they moved toward uncovering the underlying structure (which is the same in both contexts).

Of course, the behaviors exhibited by the students may also be related to similar changes in the T/R's behaviors (see Warner, Schorr, Arias and Endaya, 2006). That is, for example, the T/R also tended to set up hypothetical situations and encourage students to set up hypothetical situations more often when they were in the outer layers and less often in the inner layers. Consequently, shifts in the behaviors may also be closely related to corresponding teacher actions. Nonethless, while it is not possible to draw general conclusions based on this limited sample, this analysis has the potential to call attention to the importance to the different behaviors that might be expected as ideas grow, when students solve problems of this type in similar learning environments.

Knowing this may help teachers to be more conscious of the value of encouraging these same types of student behaviors (e.g. student-to-student questions, setting up hypothetical situations, student explanations and justifications) in their classrooms. Further research would be needed to see if these specific student behaviors are associated with similar layers of the growth of an idea for other students, exploring different types of tasks, focusing on different mathematical topics, in different learning environments, as well.

References

Carpenter, T.P., & Lehrer, R. (1999). Teaching and learning mathematics with understanding. In E. Fennema & T.A. Romberg (Eds.), *Mathematic classrooms that promote understanding* (pp. 19-32). Hillside, NJ: Lawrence Erlbaum.

Cobb, P. (2000). From representations to symbolizing: introductory comments on semiotics and mathematical learning. In P. Cobb, E. Yackel, & K. McClain (Eds.) *Symbolizing and communicating in mathematics classrooms*. Lawrence Erlbaum, Hillside NJ. (pp. 17-36).

Cobb, P., Boufi, A., McClain, K., & Whitenack, J. (1997). Reflective discourse and collective reflection. *Journal of Research in Mathematics Education, 28*, 258-277.

Doerfler, W. (2000). Means for meaning. In P. Cobb, E. Yackel, & K. McClain (Eds.) *Symbolizing and communicating in mathematics classrooms.* Lawrence Erlbaum, Hillside NJ.w (pp. 99-132).

Kaput, J. (1999). Representations, inscriptions, descriptions and learning: A kaleidoscope of windows. In Goldin & Janvier (Eds.) *Journal of Mathematical Behavior: Representations and the Psychology of Mathematics Education: Part II, 17(2),* 265–281.

Kiczek, R.D., Maher, C.A. & Speiser, R. (2001). Tracing the origins and extensions of Michael's representations. In A. A. Cuoco & F. R. Curcio (Eds.), *The Roles of Representation in School Mathematics: 2001 Yearbook*, Chapter 16, (pp. 201-214). Reston, Virginia: National Council of Teachers of Mathematics.

LeCompte, M.D. & Goetz, J.P. (1982). Problems of reliability and validity in ethnographic research. *Review of Educational Researcher, 13* (5), 20-30.

Maher, C.A. (2002). How students structure their own investigations and educate us: What we have learned from a 14 year study. In A.D. Cockburn & E. Nardi (Eds.), *26th Annual Conference of the International Group for the Psychology of Mathematics Education: Learning from Learners*, *4. Norwich, UK*, 31-46.

Maher, C.A. & Speiser, B. (1997). How far can you go with block towers? In E. Penkonen (Ed.), *Proceedings of the Twenty-First Conference of the International Group for the Psychology of Mathematics Education, Helsinki*, *4,* 174-181.

Miles, M.B. & Huberman, A.M. (1994). *Qualitative data analysis,* Chapter 4. Thousand Oaks: Sage.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

National Research Council, Mathematical Sciences Education Board. (1989)*. Everybody counts: A report to the Nation on the Future of Mathematics Education.* Washington D.C.: National Academy Press.

Pirie, S.E.B. (1988). Understanding-Instrumental, reelational, formal, intuitive…, How can we know?, *For the Learning of Mathematics 8(3*), 2-6

Pirie, S.E.B. and Kieren, T.E. (1991). Folding back: Dynamics in the growth of mathematical understanding. *In F. Furinghetti(ed.), Proceedings Fifteenth Psycology of Mathematics Education Conference, Assisi.*

Pirie, S. E. B. & Kieren, T. E. (1992a). Watching Sandy's understanding grow. *The Journal of Mathematical Behavior, 11* (3), 243-257

Pirie, S. E.B. & Kieren, T. E. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics, 26,* 165-190.

Powell, A.B., Francisco, J.M. & Maher, C.A. (2003) An analytical model for Studying the Development of Learners' Mathematical Ideas using Videotape Data. *Journal of Mathematical Behavior, 22,* 405-435.

Powell, A.B. & Maher, C.A. (2003). Heuristics of twelfth graders building isomorphisms. In N.A. Pateman, B.J. Dougherty & J. Zillox (eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education held jointly with the 25th Conference of PME-NA, Honolulu, Hawaii, 4*, 23-30.

Principles and Standards for School Mathematics. *Reston, VA: National Council of Teachers of Mathematics, 2000.*

Schorr, R.Y. (2003). Motion, speed, and other ideas that "should be put in books". *Journal of Mathematical Behavior. 22*, (4), 467-479.

Sfard, A. (2000). Symbolizing mathematical reality into being—or how mathematical discourse and mathematical objects create each other. In P. Cobb, E. Yackel, & K. McClain (Eds.) *Symbolizing and communicating in mathematics classrooms.* Lawrence Erlbaum, Hillside NJ. (pp. 37-98).

Shafer, M.K., & Romberg, T. (1999). Assessment in classrooms that promote understanding. In E. Fennema & T.A. Romberg (Eds.), *Mathematics classrooms That Promote Understanding* (pp. 159-184). Lawrence Erlbaum. Hillside NJ.

Speiser, B. (2002) How does building arguments relate to the development of understanding? A response to the last three papers. *Journal of Mathematical Behavior, 21,* 491-497.

Speiser, R. & Walter, C. (1997). Performing algebra: Emergent discourse in a fifth-grade classroom. *Journal of Mathematical Behavior, 16*(1), 39-49.

Speiser, B. Walter, C. & Maher, C.A. (2003). Representing Motion: An Experiment in Learning. *Journal of Mathematical Behavior, v. 22*, pp. 1-35.

Stein, M.K., Smith, M.S., Henningsen, M.A. & Silver, E.A. (2000). Implementing standards-based mathematics instruction: a casebook for professional development. Teachers College, Columbia University.

Uptegrove, E. B. and Maher, C. A. (2004). Students building isomorphisms. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the Twenty-Eighth Conference of the International Group for the Psychology of Mathematics Education*, *Bergen, Norway*, 353-360.

Walter, J.G. (2004). Tracing mathematical inquiry: High school students mathematizing a shell. *Dissertation Abstracts International*, *64* (12), 364A. (UMI No. 3117648)

Warner, L. B. (2005) Behaviors that indicate mathematical flexible thought. (Doctoral dissertation, Rutgers, The State University of New Jersey, 2005). *Dissertation Abstracts International, 66/01, 123*.

Warner, L.B., Coppolo Jr., J. & Davis, G.E. (2002). Flexible mathematical thought. In A.D. Cockburn & E. Nardi (Eds.), *26th Annual Conference of the International Group for the Psychology of Mathematics Education: Vol. 4. Learning from Learners.* (31-46), Norwich, UK: School of Education and Professional Development University of East Anglia.

Warner L.B., Davis, G.E., Alcock L.J., & Coppolo J. (2002). Flexible mathematical thinking and multiple representations in middle school mathematics. *Mediterranean Journal for Research in Mathematics Education, 1(2), 37-61*.

Warner, L.B. & Schorr, R. Y. (2004). From primitive knowing to formalising: The role of student-to-student questioning in the development of mathematical understanding. In D. McDougall & J.A. Ross (Eds.) *Proceedings of the Twenty-Sixth Annual Meeting of the North American Chapter of the*

International Group for the Psychology of Mathematics Education: Building Connections Between Communities, Toronto, Ontario, *2,* 429-437.

Warner, L.B., Schorr, R.Y., Arias, C., Sanchez, L. & Endaya, F. (2006) How do changes in teacher behaviors impact the linking of representations and generalizations in students? *Proceedings of the Twenty-Eighth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Merida, Mexico,* vol. 2, 647-652.

Whitenack, J. W. & Knipping, N. (2002) Argumentation, instructional design theory and students' mathematical learning: a case for coordinating interpretative lenses. *Journal of Mathematical Behavior, 21*, 441-457.

Yackel, E. (2002) What we can learn from anlayzing the teacher's role in collective argumentation. *Journal of Mathematical Behavior, 21,* 423-440.