EXTENDING AND REFINING MODELS FOR THINKING ABOUT DIVISION OF FRACTIONS

Sylvia Bulgar	Roberta Y. Schorr	Lisa B. Warner
sbulgar@rider.edu	schorr@rci.rutgers.edu	LWarnerB@aol.com
Rider University	Rutgers University	Rutgers University

Many students experience great difficulty when studying topics related to fractions, especially division of fractions. One explanation for this may be that learning how to divide fractions is often taught devoid of meaning. The lack of sense making in carrying out algorithms without making connections to concrete or other types of representations contributes to the inability of students to use previously taught algorithms to solve new problems, especially after long periods of time have elapsed. In this paper we explore the flexibility and durability of knowledge that students acquire when they study this topic in a way that encourages understanding.

INTRODUCTION

The difficulties that many students have experienced while solving problems involving fractions have been well documented (cf. Tzur, 1999; Davis, Hunting, and, Pearn, 1993; Davis, Alston, and Maher, 1991; Steffe, von Glasersfeld, Richards and Cobb, 1983; Steffe, Cobb and von Glasersfeld, 1988). It is therefore particularly important to find ways to help students overcome these difficulties. Fortunately, many researchers have also documented instances in which students have successfully been able to build ideas relating to fractions (c.f. Steencken, 2001; Steencken and Maher, 2002; Ma, 1999; Cobb, Boufi, McClain and Whitenack, 1997, Kamii and Dominick, 1997). In particular, Bulgar (2002; 2003a; 2003b; Bulgar, Schorr & Maher, 2002) [1] reports on the conceptual development of ideas relating to division of fractions amongst fourth grade students participating in a teaching experiment. Further, Bulgar reports that when this teaching experiment was replicated as part of the regular teaching practice in another classroom (her own), similar outcomes were achieved.

In this paper we report on the latter group of students, with a particular focus on how they extended, modified, revised and ultimately generalized their ideas relating to division of fractions during the following school year. This is done with a focus on mathematical flexibility, and the nature of the models that were used, and how they evolved during the following school year. In particular, we focus on how students initially used continuous linear models, how these models evolved into discrete area models and how these students moved easily back to linear models when they found them to be more appropriate.

THEORETICAL FRAMEWORK

Our framework for analysis is based primarily upon a models and modeling perspective with a specific focus on the durability and flexibility of the models that are built over time. Briefly stated (see Schorr & Koellner-Clark, 2003, for a more complete description) a model can be considered to be a way to describe, explain, construct or manipulate an experience, or a coordinated variety of experiences. A person interprets a situation by mapping it into his or her own internal model, which helps him or her to make sense of the situation. Once the situation has been interpreted into the internal model, transformations, modifications, extensions, or revisions within the model can occur, which in turn provide the means by which the person can make predictions, descriptions, or explanations, for use in the situation at hand. Models help us to organize relevant information and consider meaningful patterns that can be used to interpret or reinterpret hypotheses about given situations or events, generate explanations of how information is related, and make decisions about how and when to use selected cues and information.

We wish to distinguish between the conceptual "models" that are embodied in the representational media that students use, and the "mental" models that reside inside the minds of learners (Lesh and Doerr, 2003) to which we refer above. In this work, we will be attending to both, with an emphasis on the nature of the mental models that are born out in the representational media that the students use, especially as evidenced in their mathematical flexibility. We document the nature of the models that the students have built in terms their "mathematical flexibility" not just during or shortly after the instruction took place, but rather over a more extended period of time. We address flexible thought in the context that follows, and because it is relevant to this study.

Carey (1991) describes flexibility by saying "… children become more flexible in their choice of solution strategy as a result of changes in their conceptual knowledge, so that they can solve problems using a variety of strategies that do not model directly the action in the problem" (p. 267). Heirdsfield (2002) notes that flexibility is the capacity of students to exhibit various invented strategies or a large repertoire of problem-solving strategies over time. She referred to the use of a single strategy consistently as inflexibility. Further, Gray and Tall (1994) describe flexible thinking in terms of an ability to move between interpreting notation as an instruction to do something (procedural use of notation) and as an object to think with and about (conceptual use of notation). Flexibility, as denoted in the work by Spiro & Jehng (1990) entails the ability to spontaneously restructure one's knowledge, in adaptive response to changing situational demands. Krutetskii (1969) characterizes flexible thinking as reversibility of thought—another much needed characteristic for students as they consider ideas related to fractions over time. Other researchers including Warner, Alcock, Coppolo & Davis (2003) and Warner and Schorr (in progress) emphasize that a critical aspect of mathematical flexibility is the ability of students to use multiple representations for the same idea, and to link, extend, and modify those representations to a broader range of situations, involving a broader range of models. Since the goal of our instruction was not simply to have students retrieve facts or procedures, or to display understanding only for very specific situations and for limited time periods, we believe that mathematical flexibility is particularly relevant, as defined by all of the researchers above.

Mathematical flexibility is particularly important if students are to use knowledge across a wide spectrum of ideas. Fosnot and Dolk (2001) note, "The generalizing across problems, across models, and across operations is at the heart of models that are tools for thinking." (p.81). They report on a class in New York City wherein a third grade teacher provided students with three different contexts that lent themselves to different models but produced the same answer. In each case the children produced different models that were closely linked to the context. Fosnot and Dolk go on to state that it is easy for students to notice that the answers are the same but that the important issue is for them to see the connections among the models to develop a generalized framework for the operations. In the work that follows, we focus on the nature and type of representations that students build, retrieve, and use over time, and how this relates to their mathematical flexibility.

The students in these studies had, essentially, used three main strategies to solve a particular series of problems (Bulgar, 2002; 2003a; 2003b; Bulgar, Schorr & Maher, 2002). There were no strategies other than these three observed in either the classroom-based studies or the teaching experiment.

These strategies consisted of the following:

- Reasoning involving natural numbers
- Reasoning involving measurement
- Reasoning involving fraction knowledge

The predominant solution method observed in the fourth grade study of the same students (see Bulgar, 2002, 2003a, 2003b) consisted of reasoning involving natural numbers. Essentially, these students built models that converted the meters to centimeters, thereby substituting the fraction division with natural number division, a topic generally prominent in fourth grade mathematics curricula in New Jersey (NJ Mathematics Coalition and NJ State Department of Education, 2002). However this solution method was seen in the work of only one fifth grader in the replicated study, and even when it did appear, there was a claim by the student that it was developed after the problem was solved using reasoning involving fraction knowledge (Bulgar, 2003a, b). All of those students in the fifth grade who drew representations, created linear models to represent the division of a piece of ribbon into various-sized bows.

METHODS AND PROCEDURES

BACKGROUND, SETTING AND SUBJECTS

The study currently addressed took place during the 2001-2002 school year, when the subjects, thirteen girls, were in sixth grade. Twelve of these students had been taught mathematics by the same teacher, the first author of this paper, during fifth grade. The students attended a small parochial school in New Jersey, which attracts children from several surrounding communities. A fundamental premise of the instructional environment was that in order to build mathematical ideas, students needed to be engaged in mathematical activities that promote understanding (Davis & Maher, 1997; Maher, 1998; Cobb, Wood, Yackel & McNeal, 1993; NCTM, 2000; Klein and Tirosh, 2000; Schorr, 2000; Schorr and Lesh, 2003). Therefore, conditions established during the fifth grade, were set up to create a classroom community in which student inquiry and discovery were of paramount importance. The classroom environment was one in which students' ideas were always respected. Students were questioned and encouraged to explain their solutions, developing their own sense of accuracy. Alternate strategies were encouraged, shared and discussed, as students were invited to discuss their thinking and to submit ideas in writing. Students were not taught algorithms. When they recognized patterns and could justify that these patterns were valid, they created generalizations, which they could apply to future problems. Questions were used to elicit explanations, to guide students towards persuasive justifications of their solutions and to redirect them when they were engaged in faulty reasoning. Justification of solutions became a part of the classroom culture.

Because essentially the same group of students who were taught by the first author in fifth grade were grouped together again in sixth grade, (Twelve of the students were from the original group and there was an addition of one new student.) and taught mathematics by the same teacher, an opportunity was presented to closely examine longitudinal development of mathematical ideas within the framework of regular teaching practice. The Tuna Sandwiches task, the problem that is the subject of this paper, was the first one assigned as these students began sixth grade.

DATA

The data examined here consist of artifacts of actual student work, which were collected over the course of approximately six weeks. Written notes from the teacher were attached to some of the work, usually in the form of questions and answers to these questions also appear in the students' writing.

TASKS

The primary task studied here, "Tuna Sandwiches", was created by the first author to be isomorphic with the problem done during the previous year called "Holiday Bows"[2], which introduces division of a natural number by a common fraction. The Tuna Sandwiches problem follows.

> Mr. Tastee's restaurant serves four different kinds of sandwiches. A *junior sandwich* contains 1/4 lb of tuna; a *regular sandwich* contains 1/3 lb of tuna; a *large sandwich* contains 1/2 lb of tuna and a *hero sandwich* contains 2/3 lb of tuna. Tuna comes in cans that are 1lb, 2 lb, 3lb and 5 lb. How many of each type of sandwich can you make from each size can? Find a clear way to record your information. You will need to write a letter to the restaurant owner, Mr. Tastee, and give him your findings.

One of the goals in creating the "Tuna Sandwiches" problem was for it to lend itself to be represented by an area model rather than a linear model, as was the case with "Holiday Bows". Fosnot and Dolk (2001) state that just because we create a problem with certain models in mind, we cannot be assured that this model will be used by students. By creating a problem that was essentially isomorphic to the "Holiday Bows" problem, (the one that was completed by both the fourth graders in the teaching experiment and the fifth graders in the regular classroom of the first author), yet embodied in a different type of representation, an area model, the notion of flexibility could be explored as well as an examination of the durability of the knowledge the students had demonstrated during the previous year.

RESULTS AND DISCUSSION

All of the sixth grade students solved the problem using the approach of reasoning involving fraction knowledge. That is, they reasoned that if a sandwich requires 1/4 of a pound of tuna, four such sandwiches could be made from every pound of tuna, so what was necessary in order to find the solution was to multiply the number of pounds of tuna in a can by four. In both the fourth grade and the fifth grade studies, dividing by the non-unit fraction, 2/3, had proven to be more problematic. One might conjecture that the linear model used by students would be more conducive to solving problems such as $2 \div 2/3$, because it is a continuous model. Yet, several fourth and fifth grade students who had used reasoning involving fraction knowledge had difficulty with this because it was arduous to give meaning to the piece that was "left over"; it was not clear how many two-thirds there were in one. One student in the fourth grade group stated the following when explaining how many bows, each 2/3 meter in length could be made from a piece of ribbon that is 2 meters long.

> Alex: There's three thirds [in one meter] so there's two-thirds and one-third and one-third that's two-thirds and you still have one two thirds left over…[*while drawing picture*] ... so then... so you only have one third so then you have to get the other third. This is two thirds so then you have two more [one] thirds left over.
>
> Jon: [*pointing to Alex's drawing*] And there are six ones [1/3] is in each, and it would be two-thirds is one [bow], two-thirds is again [a bow] and two [one] thirds left.
>
> Alex: I think it's 4 [bows].

Alex looked at the two one-meter parts of his two-meter ribbon as two discrete entities. He did not seem to realize that the two one-third meter pieces that remained at the end of each meter could be used to make another bow. Although all of the fifth grade students eventually were able to find out how many bows, each 2/3-meter in length could be made from the various lengths of ribbon, they also had greater difficulty with this set of problems than they had when dividing by the unit fractions.

None of the thirteen sixth graders used a linear model to solve the Tuna Sandwich Problem. Ten of the thirteen students actually drew area models to represent their solutions and three merely explained their thinking without referring to a model. It is interesting to note that each of these area models included discrete drawings for each pound of tuna. One would think that the problems involving the hero sandwiches, those which each required 2/3 lb. of tuna, would be more difficult to solve when using discrete area models. Yet, there was no mention of greater difficulty. In fact, several students stated that each one-pound of tuna would yield one and one-half hero sandwiches. It appeared that the shift in unit was made seamlessly. One-third pound of tuna was recognized to be one half the quantity needed to make a hero sandwich, which required two-thirds pound of tuna.

Though they were not asked to do so, most of the sixth graders spontaneously formed some kind of graphic organizer to structure their results. Seven of the thirteen students formed a matrix indicating the amount of tuna required (for each sandwich) as one dimension and the different-sized cans of tuna as the other dimension. Four of the students indicated their solutions in an organized listing. One of these students had both an organized listing and a matrix.

Since the students specified their solutions using reasoning involving fraction knowledge by looking first at how many sandwiches of each type could be made from a one pound can of tuna, it is interesting to note that very few used proportional reasoning, using multiplicative structures to arrive at solutions involving multiple-pound cans of tuna. Most used additive structures. Stephanie begins by alluding to proportional reasoning when she writes the following as she explains her solutions for finding out how many regular sandwiches, those requiring 1/3 pound of tuna, could be made from each of the various sized cans.

> Stephanie: [sic] You can only make 3 sand. With one lb of tuna because 3 thirds make 1. (3/3=1) With one more lb of tuna (2lb) you can make twice as many sand. So you have 6 sand. With 3 lb of tuna you can make 3 more sand. (9 altogether) because you have one more lb of tuna which make 3 sand. Because 3 thirds (3/3) =1. Now with 5 lb. you add not 3 sand. But 6 because it is not 4 lb, but 5 lb of tuna.

Stephanie seems to be going back and forth between multiplicative ideas and additive ones, adding on multiples of three sandwiches. When Stephanie explains her solution to the hero sandwich problem, the one involving division by a non-unit fraction, she states the following.

> Stephanie: [sic] So with a 1 lb can you can make 1 sand. and a 1/2 of another because it is 2/3 of a lb of tuna [required for each hero sandwich] so you have 2/3 left which is 1/3 left which is 1/2 of 2/3. A 2 lb can of tuna you can make 3 sand. easily and the excess is 1/3 from both so that makes 3… Now for a 5 lb. can you can make 6 1/2 sand. because you can make 5 easily and 2 1/2 more with the extra of each lb.

Though Stephanie's solution of 6 1/2 sandwiches is not consistent with her explanation, she has demonstrated an understanding that 1/3 of a pound of tuna represents 1/2 of a hero sandwich, an idea that students had more difficulty understanding the previous year when they worked with the linear model suggested by the Holiday Bows problem. It would appear that she is first counting the complete sandwiches that can be made from each pound, the ones she refers to as being made "easily", and then is gathering up the remaining 1/3 pounds from each can to combine them in order to make additional sandwiches. This kind of thinking was also observed in the representations of other students, such as Gabriella, Lynn, Amy, Sarah and Bea, who drew connecting lines to the "leftover" one-third pound of tuna in each representation of a one-pound can.

After completing a lengthy explanation of her solutions, Eve wrote the following reflection on her work.

> Eve: P.S. When I was figuring this out for you I noticed something interesting. I noticed [sic] That by the junior sandwich (1/4 lb.) you added 4 by every can of tuna. This is because every time the can get bigger by 1 lb (from which you can make 4 sandwiches) so you just add another 4 and the 5 lb., it is 2 more lbs. So you add 8 instead of 4.

Though Eve used reasoning involving fractional knowledge, she applied additive reasoning to get the solutions.

Sarah used multiplicative reasoning in finding the solutions. She wrote the following.

> Sarah: [sic] Out of 3 pound you can make 12 junior. There is 4 in each and 4 x 3 = 12.

Sarah included a diagram of 3 circles divided into four sections or fourths. She numbered the sections from one to twelve. She used this structure for all of her solutions.

Gabriella also used multiplicative reasoning. She drew five circles, divided them in half vertically and stated the following.

> Gabriella: [sic] How much large sandwiches can you make from 5 pounds. Let's try those imaginary pounds [her drawings]. Well 2 in each of the 5 pounds 5x2 = 10!

In the summative class discussion of the Tuna Sandwiches Problem, students talked about the problem and how it was just like the problem they had done the previous year called "Holiday Bows". Those who did not recognize it at first agreed when their peers noted the isomorphism. They recognized that the problem required division of fractions and easily explained their solutions using symbolic notation. For example, when summarizing that three hero sandwiches could be made from two pounds of tuna, they were able to create the number sentence, 2÷2/3 =3. Some of the number sentences that the students provided were recorded on an overhead projector transparency. These number sentences are seen as solutions representing conceptual understanding derived from the use of student-generated models, rather than as algorithmic answers. Once the students agreed that they had solved these problems involving division of fractions, they were assigned numerical problems, one at a time. The first problem was 2 ÷ 3/4. They were told to build a model to solve the problem and to explain how the model can be used to find the solution. Some (Michelle, Amy and Rose for example) wrote the problem as "How many 3/4's are in 2?" This would indicate an understanding of the meaning of division. Subsequent to providing solutions for this problem, students worked on the problem, 5/8 ÷ 2 1/2. What is interesting to note here is that when drawing models to solve these problems, students invariably went back to linear representations. Many referred specifically to Cuisenaire Rods® when they discussed their linear models. They had worked with these materials early in fifth grade to build basic concepts about fractions. The activities in which they were engaged using these materials were modeled after those used and documented in another study (Steencken, 2001).

CONCLUSIONS

Students in the sixth grade were able to retrieve ideas they had built about division of fractions during the previous school year, and these ideas were used and extended appropriately. Many students demonstrated flexible thought in the way they indicated their grasp of division of fractions and extended their understanding to more complex division of fractions problems. When first confronted with a task involving division of a natural number by a fraction in fifth grade, they made use of linear continuous models. Our results show that when a similar problem was given to the same students a year later, one that lent itself to an area model, students demonstrated flexible thinking in their ability to seamlessly move to a discrete area model and durability of the ideas they built the previous year in their ability to effortlessly move from linear models to area models and back to linear models as needed. Many recognized and verbalized that the Tuna Sandwiches Problem was "the same" as the Holiday Bows Problem. They revealed their flexible thinking in their ability to use a variety of representations for the same idea, division of fractions, and to link, extend and modify those representations to a variety of situations (Warner, Alcock, Coppolo, 2003; Warner & Schorr, in progress). They moved easily back and forth between area models and linear models as they worked on contextual tasks and used models to solve numerical problems. This is significant because as Fosnot and Dolk (1991) indicate, models represent strategies used to solve problems and thereby develop into mathematical tools. Generalization is characteristic of this development.

ENDNOTES

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[2] For a full description of this task and results see Bulgar, 2002; Bulgar, 2003a; Bulgar, 2003b; Bulgar, Schorr & Maher, 2002.

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