HOW DO CHANGES IN TEACHER BEHAVIORS IMPACT THE LINKING OF REPRESENTATIONS AND GENERALIZATIONS IN STUDENTS?

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In the present study, we focus on the ways in which five urban middle school teachers interacted with their students in order to help them build representational fluency, with a particular emphasis on the ways in which the students moved toward increasingly abstract representations and generalizations. We document advances or changes in students' representational fluency, or facility with generalizations and abstractions, how teacher interventions impact this, and how the corresponding changes in students impacts teachers.

Introduction and Theoretical Framework

This paper addresses one component of a five year longitudinal study in which Rutgers University researchers partner with teachers, teacher educators, and administrators in the Newark Public Schools[1] in order to help students in grades kindergarten through eight develop a deeper and more meaningful understanding of mathematical concepts. In the current study, we focus on the ways in which teachers interact with students as they build, extend, link, and refine representations. In particular, we extend prior research (e.g. Schorr, 2004; Schorr & Lesh, 2003) in which the reciprocal relationship involved between teacher behaviors and actions on student behaviors and actions, and vice versa, is documented. This paper extends that work to include student development of representational fluency leading toward generalizations and abstractions.

Schorr, 2004; Schorr & Lesh, 2003; Schorr, Warner, Gearhart & Samuels, (in press) report that as teachers develop new knowledge, they notice new things about their students as well as their teaching practices. This, in turn, causes them to revise their approaches to teaching. As this happens, their students also change, thus resulting in further changes on the part of the teacher. Our central hypothesis is that when teachers encourage students to make sense of different types of representations (both their own and their peers) in a way that elicits meaning and sense making, students move toward increasingly abstract representations and generalizations. As this happens, teachers modify their behaviors, which elicits further changes in students' movement toward generalization and abstraction. Before continuing, we define, in a broad sense, the ways in which we use the words representation, abstraction, and generalization.

Representations: Broadly defined, the term representation "refers both to process and to product—in other words, to the act of capturing a mathematical concept or relationship in some form and to the form itself" (p. 67, NCTM, 2000). Some well recognized forms of representation include pictures, tables, graphs, diagrams, and strings of symbols. These forms of representation have a long established and highly stable place in school mathematics, and in many cases, are often taught for their own sake (NCTM, 2000; Kaput & Schorr, in press). Representations also take place in the context of spoken, kinesthetic, physical or cybernetic formats (Kaput & Schorr, in press), and can include mental representations, although it often happens that one makes use of physical materials in the process. Goldin & Shteingold (2001) offer a broad view of representation by describing it as sign or a configuration of signs, characters or objects. They note, "the important thing is that it can stand for (symbolize, depict, encode or represent) something other than itself" (p. 5). Speiser & Walter (1997) add a very important dimension in their description of a representation as "a presentation, perhaps to oneself, as part of an ongoing thought; or perhaps to others, as part of an emerging discourse."

The development and use of various representations, whether verbal or written, are important for many reasons. For example, Lesh (1998) underscores the importance of representations that are well known to students by stating that students can build and use sophisticated constructs when these understandings are grounded in familiar modes of representation. Yackel (2002) shows how actions, diagrams and notation, as well as verbal statements, can function as an important component of argumentation. Kaput (1999) notes that when students use their own representations to reason and argue, link their own representations to abstract representations in an effort to justify ideas and use other representations to justify their generalizations, they develop a sense of ownership of the mathematical ideas.

Abstraction and Generalization: Abstraction and generalization are two of the most important aspects of mathematical thought (see Bochner, 1966, as cited in Kaput & Schorr, in press). Abstraction, which the NCTM Standards (2000) refers to "the stripping away by symbolization of some features of a problem that are not necessary for analysis, allowing the 'naked symbols' to be operated on easily. In many ways, this fact lies behind the power of mathematical applications and modeling" (p. 69, NCTM, 2000).

Kaput (1999) describes generalization as "deliberately extending the range of reasoning or communication beyond the case or cases considered, explicitly identifying and exposing commonality across cases, or lifting the reasoning or communication to a level where the focus is no longer on the cases or situation themselves but rather on the patterns, procedures, structures and the relationships across and among them" (p. 136). Mason (1998) notes that generality is at the heart of all mathematics and states, "Explicitly getting students to specialise from generalities, and to generalize from particular cases supports them in processes which are often left below the surface of awareness because they are so fundamental, so important" (p.3). **Teacher actions and behaviors leading toward abstraction and/or generalization:** Mason (1998) notes that some types of teacher questions can lead students toward or away from generalizing. In particular, he notes that teacher questions can be used as springboards upon which students simply try to guess what is in the teacher's mind while at other times, teachers can ask questions in a manner that genuinely supports students as they learn to analyze, generalize, and justify conclusions. Martino & Maher (1999) note that, "teacher questioning that is directed to probe for student justification of solutions has the effect of stimulating students to re-examine their original solution in an attempt to offer a more adequate explanation, justification and/or generalization" (p. 75). Questioning, however, is not the only mechanism by which teachers can help students to link representations, generalize or abstract ideas. Other researchers (e.g. Speiser & Walter, 1997; Warner & Schorr, 2004) note that when used appropriately, teachers can use students' own representations to focus emergent explanations and justifications. One of the practical implications is that students need to be given the opportunity to construct their own representations of mathematical concepts and relationships, as well as the time and support needed to develop and use symbolic representations. Beyond that, Warner & Schorr, (2004); and Warner, Schorr, Gearhart & Samuels, (2005) note several other types of teacher behaviors that can impact students' ability to link representations and move toward increasingly abstract representations and generalization. These include, for example, highlighting student ideas, encouraging students to build on their own and/or others' ideas, and setting up hypothetical problem situations based on an existing problem.

The research questions that will be addressed in this paper are: How do teachers interact with their students as they build, modify, and link representations and move toward abstraction and/or generalization; and, how do teacher actions impact student actions, and vice versa?

Methodology

Background: The research reported on in this study takes place in Newark, the largest urban school district in the state of New Jersey. It is based upon a pilot project in which the work of several middle school teachers was examined over the course of several years (Schorr, Warner, Gearhart, & Samuels, in press; Warner, Schorr, Gearhart & Samuels, 2005). The teachers who were involved in this study participated in professional development sessions with researchers both at the University and within the context of their own classrooms. During all aspects of the professional development, teachers had the opportunity to consider, amongst other things, mathematical ideas; classroom implementation strategies; student knowledge development; and, building a classroom culture in which proof, justification, sense making, and high cognitive demand are the norm (Stein, et al, 2000). Two key components of the professional development involved weekly meetings at a university and on-site support to the teachers as they implemented project activities in their classrooms (see Schorr et. al. in press, for a more complete description).

Subjects: The subjects for this research were five middle school teachers (teachers of grades 6, 7, or 8). In all instances, the teachers had participated in at least three months of professional development. See Table 1 for details[2]. The reader will notice that two teachers were relatively new to the project while the other three were involved for at least one full year.

Teacher	Total length of time teacher was involved in study	Grade level taught during the study	Duration of time the observations took place	# of actual minutes of videotape used for analysis[3]	# of sessions @ # of minutes each
Ms. A	3 months	6th	3 months	450 minutes	5 @ 90 min
Ms. J	7 months	7th	3 months	540 minutes	9 @ 60 min.
Mr. C	1.5 years	8th	3 months	810 minutes	9 @ 90 min.
Ms. E	2.5 years	7th & 8th	1.5 years	1260 minutes	9 @ 100 min. & 4 @ 90 min.
Mr. R	3 years	8th	6 months	960 minutes	5 @ 120 min. & 5 @ 90 min.

Table 1: Information for each Teacher

Data: The data that forms the basis for this study consists of at least three months of actual classroom sessions (University students videotaped the sessions). In each case, at least two video cameras captured different views of the teacher, students' group work, students' presentations, etc. In addition, the teachers were asked to provide written reflections immediately after each session. All student work was collected and descriptive field notes were compiled.

Analysis: The classroom episodes were analyzed (using observations, field notes and videotapes). We identified episodes by students' movement toward abstraction and/or generalization. These were then summarized, transcribed, and coded for instances of behaviors that appeared to impact the growth of student representations, abstractions, and generalizations. These were checked and verified by at least two researchers.

Results

The results will be presented according to several overarching themes that emerged from close analysis of the data. The themes represent some of the many types of interactions that took place between teachers and students regarding the development and use of representations, especially those that resulted in abstractions and generalizations. We note that while all teachers were unique in their development, there were many instances in which one or more exhibited similar types of behaviors. These are described in the sections that follow.

Overall reactions of teachers: In the early months of our observations, all five teachers allowed their students to solve complex problems in small groups. The two teachers (Ms. A & Ms. J) that had joined the project later on, appeared to be uncomfortable spending too much time on any one task—time that would be needed for students to move beyond their initial conceptualizations toward abstractions and generalizations. Ms. A actually noted that she had trouble understanding where the students' representations might lead. During the last month however, Ms. J became more comfortable with providing additional time for the students to explore a task, although she noted that she wasn't sure what to look for or to ask.

We also found, initially, that while the teachers allowed students to have extended periods of time to work on tasks and would go from group to group observing students and questioning them about their work, some of the teachers (Ms. E & Ms. J in particular) spent very little time with each group and tended to ask questions based upon their own thinking rather than on their student's. Over time, however, this decreased for all teachers. Further, the number of instances in which the teachers interrupted or interfered with a student's way of thinking decreased as well. Not coincidently, teachers appeared more able to follow students' ideas and build upon their thinking. In all cases there was an increase in teachers encouraging students to explain, question, use, justify, and build on their own or others' representations, and a corresponding increase in the students' willingness to actually talk about their ideas with their peers.

As the weeks progressed, Ms. A, Mr. C, Ms. E & Mr. R began to pose hypothetical problem situations (e.g. "What if…" scenarios) to the students as a way to stimulate movement to generalization. The students responded by posing their own hypothetical situations. The instances of this increased over the course of the observations (for a more complete description of what is meant by "raising hypothetical problem situations" see Warner, Coppolo & Davis, 2002). Ms. J. also began to raise hypothetical problem situations, but only at the end of the observation period. Several of her students began to raise hypothetical problem situations shortly thereafter. This appeared to contribute to students' movement toward abstraction and generalization in several cases for Ms. J & Ms. A and even more so for Ms. E, Mr. C & Mr. R.

Linking representations and the movement toward abstraction and generalization: A central tenant of the professional development involved encouraging teachers to build, extend, and link the representations that they had built for a particular idea. Early on in the study, several of the teachers expressed a desire to have their own students do the same. Mr. R, Mr. C and Ms. E all noted that by doing this, their students would most likely be able to formulate more succinct and convincing justifications. Early on in the observation period, Mr. C. noted: "I also like to investigate how students ideas and/or depictions of their ideas interrelate… I asked them, when all the groups were done presenting, was if they saw any relationships between each other's models, if so, what were they and if any one model would be sufficient to answer the question. If every student sees every other student's methodology and has it explained by the creator then solving this problem and other subsequent problems will become easier" (personal communication, March 24, 2005). In a similar vein, Ms. E writes:

> I had an aha moment today while watching … [videotapes of another class]. I think I know what I need to be doing when the students are starting to generalize….In the past, I thought if one student found the rule or the formula or the shortcut (whatever the students want to call it), that that student should present and everyone in the classroom will get it by listening from that one student. I realized … that the students do not internalize the rule if they do not find it themselves through exploration. The fact that only one or two students have generalized says to me that the class in general is not ready to see the formula. Thus, I

should try to ask the students to "link the representations". This is what you [referring to the researchers] told me before, but like the students I have not internalized what you mean by that. The goal is not for one student to find a formula and to present it to the class, but for all the students to try to link different kinds of representations such as diagrams, reports, mathematical representations, and so on (personal communication, September 25, 2005).

As Mr. R, Mr. C & Ms. E discussed this idea with the researchers, and each other, there was an increase in their students linking representations and movement toward generalization.

Encouraging students to justify their generalizations: Early in the study, Mr. R wrote, "Looking back… I see now that a lot of my students did get it, but I didn't allow them to explore what they had....If I had allowed them to vocalize their solutions and come to some real ownership, then several months later when asked to express these ideas in a different context, they'd be able to apply it" (personal communication, May 24, 2004). In the earlier part of our observations, Mr. R, Mr. C & Ms. E began to encourage the students to continue to explore even after they developed a generalized solution to the problem. They also encouraged the students to create more than one type of generalization or express their generality in different formats (when appropriate). Mr. R and Mr. C also asked students to justify their generalizations or general statements and look at the relationships between them. At the end of the observation period, the students in Mr. R's classroom appeared to be dissatisfied unless they could generate multiple generalizations, and convince each other of the efficacy of each. Indeed, at a certain point in time, Mr. R. no longer had to instigate these discussions. The students spontaneously asked each other to explain and link their generalizations and justifications.

Allowing students to assume ownership of their ideas: We noticed that at the beginning of the observation period, students in all classes sought their teacher's approval for their ideas, questions and solutions. Mr. R and Ms. E dealt with this by encouraging the students to challenge, question, and direct their responses to each other. They shared that they often used cues determined by "reading" the body language, facial expressions and/or gestures of the students as a way to encourage them to talk directly to each other. Their students did talk directly to each other, but this did not happen consistently, except in the case of Mr. R, where by the middle of the observation period, his students consistently directed their comments to each other, without using him as an intermediary. Further, his students challenged each other to explain all representations and ideas, demanded that these be connected, and felt that they could and should try, when possible, to build generalized formats that connected solutions to the problem and hypothetical extensions of the problem. Mr. R. noted that, "the questions that we ask serve as models for the students. They begin to ask the same types of questions of each other as we do of them. Students are no longer satisfied with the answers to the problems that are posed, rather they seek proof and justification and question until they are satisfied. As students began to question each other with skill, tasks took on a whole new life." (personal communication, June 7, 2004). The other two teachers (who were newer to the project) did not encourage this to the same extent, and their students did not tend to "talk" directly to each other.

Finally, we note that as Lannin, Barker & Townshend (in press) point out there can be great benefit in having students investigate their own errors as a way to deepen their understanding and ability to generalize. All five teachers attempted to use errors as a way to probe students, and they also tried to minimize the number of instances in which they actually "told" students that they were incorrect. Rather, they tried to let students investigate their errors by encouraging them to justify their solutions. All of the teachers noted the importance of not "telling" students the answer, but many had difficulty in actually doing this. There were many occasions, particularly at the beginning of the observations, where they used facial expressions, intonations, or body gestures, to let students know exactly what they were thinking, even when they did not directly tell students the answers. Once again, Mr. R. was unique in that by the middle of the observation period, his words, voice, posture, and facial expressions suggested neutrality.

Conclusion

Teachers in our study appeared to proceed through several steps as they helped students link representations and move toward abstraction and generalization. In the beginning, many did not see the advantage in having students engage in lengthy explorations wherein such activity could take place. Once they recognized the importance, they had difficulty in actually making it happen. Some would ask students questions that they felt would stimulate representational fluency, generalization, and abstraction, but upon reflection, noticed that their questions were often based on their own preconceived notions about what a "correct" representation should be. Some teachers used subtle cues (i.e. facial expressions), while others actually felt the need to more directly steer students in their problem solving process. As teachers encouraged students to, for instance, investigate hypothetical problem situations, link representations, etc., we found that the students actually started demanding this themselves. This, in turn, served as a feedback mechanism for the teachers, wherein they felt a sense of affirmation for their new practices. Ms. E, one of the teachers involved in the study, and an author of this paper, best sums it up by saying, "It is when the teacher models certain behaviors such as requiring students to justify their answers, pose hypothetical situations, provide opportunities for them to generalize and abstract representations, that students emulate them to the extent of even owning these behaviors after several times of being on the receiving end of the experience…the possibilities of exploring and learning mathematically challenging ideas are extended way beyond the finding of a generalization that is derived from the problem initially posed" (personal communication, January 29, 2006).

Endnotes

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2. Multiple camera views do not add to the number of minutes.

3. The number of minutes varies by school and year.

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