

FROM PRIMITIVE KNOWING TO FORMALISING: THE ROLE OF STUDENT-TO-STUDENT QUESTIONING IN THE DEVELOPMENT OF MATHEMATICAL UNDERSTANDING

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In this paper, we examine the development of inner city middle school students' ideas and the student-to-student interactions and questions that contribute to this development within the context of the Pirie/Kieren model. We analyze data collected from an inquiry oriented, problem based mathematics class in which students were repeatedly challenged to explain their thinking to each other, and defend and justify all solutions. In this instance, we document how one student was able to move from primitive knowing to formalising. Further, we note that this student (and her classmates) were able to use this knowledge several months later when solving a structurally similar problem.

OBJECTIVES/PURPOSES

Prompting students to talk about mathematics is an important goal of education (NCTM 2000; Sfard, 2000; Dorfler, 2000; Cobb, Boufi, McClain, and Whiteneck, 1997). Cobb, (2000) notes that student exchanges with others can constitute a significant mechanism by which they modify their mathematical meanings. Carpenter and Lehrer, (1999) state that "the ability to communicate or articulate one's ideas is an important goal of education, and it also is a benchmark of understanding." (p. 22) Researchers such as those cited above (and others, see for example, Schorr, 2003; Maher, 2002; Shafer and Romberg, 1999) maintain that it is important to provide students with opportunities to discuss their ideas with each other, defend and justify their thinking both orally and in writing and reflect upon the mathematical thinking of others. One important component of this involves students' questioning the mathematical thinking of their peers. This report focuses on the impact of student questioning on the development of mathematical thinking. We do this within the context of the Pirie/Kieren theory for the growth of mathematical understanding (Pirie and Kieren, 1994).

Our central premise is that when students have the opportunity to question each other about their mathematical ideas, both the questioner and the questioned have the opportunity to move beyond their initial or intermediate conceptualizations about the mathematical ideas involved. As students reflect on their own thinking in response to questions that are posed by their peers they have the opportunity to revise, refine, and extend their ways of thinking about the mathematics. As they do this, their earlier conceptualizations and representations become increasingly refined and linked. We stress the role of representations in this dynamic since "the ways in which mathematical ideas are represented is fundamental to how people can understand and use those ideas." (NCTM, p.67) In this paper, we will trace the development of ideas (using the Pirie/Kieren model) and the student-to-student interactions and questions that contribute to this development.

THEORETICAL FRAMEWORK

In 1988, Pirie discussed the idea of using categories in characterizing the growth of understanding, observing understanding as a whole dynamic process and not as a single or multi-valued acquisition, nor as a linear combination of knowledge categories. In 1994, Pirie & Kieren described eight potential layers or distinct modes within the growth of understanding for a specific person, on any specific topic. The inner-most layer, called primitive knowing, is what a person can do initially and is the starting place for the growth of any particular mathematical understanding. When a person is doing something to get the idea

of what the concept is, he/she is working in the image making layer. A person working in this layer is tied to the action or tied to the doing. Working in the image having layer is when one reaches a “don’t need boundary”, where he/she is no longer tied to the action or the doing. When a person “...can manipulate or combine aspects of ones images to construct context specific, relevant properties” and ask themselves how these images are connected, one is property noticing (Pirie & Kieren, 1994, p.66). When one no longer needs to talk specific and can make a general statement, he/she is formalising. When formalising, “the person abstracts a method or common quality from the previous image dependent know how which characterizes his/her noticed properties” (Pirie & Kieren, 1994, p. 66). This theory is a way to explain understanding and is a useful tool for understanding how understanding grows. The structure of the theory is non-linear, repeating itself with many layers wrapped around.

.In 2003, Warner, Alcock, Coppolo & Davis linked this theory to specific behaviors that indicate mathematical flexible thought. Briefly stated, a person exhibiting mathematical flexibility may be characterized as one who displays some or all of the following behaviors: interpretation of their own or someone else’s idea (e.g. through questioning it and thus showing it to be valid or invalid; through using, reorganizing or building on it); use of the same idea in different contexts; sensible raising of hypothetical problem situations based on an existing problem: creating “What if...?” scenarios; use of multiple representations for the same idea; connecting representations (Warner, Coppolo & Davis, 2002). In this study, we will illustrate movement through the first six layers (described above) as we focus on how the transitions from one layer to the next occurred in association with student-to-student interactions and questioning. We will also highlight how these student-to student interactions and questions move students to new representations and the linking of these representations.

METHODS

The study took place over the course of 8 months, which involved two visits a week (50 minutes each session), for the first two months, and 3 to 6 visits a month for the remaining 6 months, in a diverse eighth grade inner city classroom with approximately 30 students. The visits were part of a professional development project in which the teacher/researcher (first author), who is a mathematics education researcher at a local university, routinely met with local teachers, planned classroom implementations, and then modeled or co-taught lessons with the teacher. After each lesson, the teacher/researcher would “debrief” with the classroom teacher and a University mathematics education professor (the second author) to discuss key ideas relating to classroom implementation, the development of mathematical ideas, and other relevant issues. During the course of the eight months, several different tasks were explored. The teacher/researcher, along with the classroom teacher encouraged the students to exchange, talk about, and represent ideas; conjecture, question, justify and defend solutions; discuss disagreements and differences; revisit ideas over time; and, generalize and extend their ideas. Generally, the students worked in groups of 3-5, and each group discussed, argued, and ultimately presented its solutions.

During each class session, two cameras captured different views of the group work, class presentations and associated audience interaction. In addition, careful field notes were taken after each session. This study focuses on 8 of the 62 videotapes generated in this manner, as the students explore variations of a task. The problem task was as follows: *John is having a Halloween party. Every person shakes hands with each person at the party once. Twenty-eight handshakes take place. How many people are at the party? Convince us.*

This particular problem entails a context that may suggest a structure that ultimately leads to a solution that is generalizable to a larger class of problems. In this case, such a solution is $[n(n-1)]/2$.

Episodes were transcribed and coded to identify critical events, which in this case were determined by student-to-student questions and/or interactions.

In the sections that follow, we examine the development of a particular student, Aiesha, by identifying student-to-student questions and/or interactions in the context of the Pirie/Kieren model for mathematical understanding.

RESULTS

MOVING FROM PRIMITIVE KNOWING TO IMAGE MAKING

Primitive knowing is the starting place for the growth of any particular mathematical understanding, what the student can do initially, with the exception of the knowledge of the topic. In this case, Aiesha begins by shaking hands with a member of her group and then moves to a picture and number representation for her idea (figure 1). She moves to the image making layer (doing something to get the idea of what the concept is), using a picture representation to construct an idea of multiplying the number of people by one less than the number of people to arrive at the number of handshakes. Every time she multiplies, however, she arrives at double the number of actual handshakes in the correct solution. At first, she doesn't notice this mistake and becomes frustrated, explaining that there is no answer. After another student shares his solution, she realizes that the answer is eight and divides her answer to an eight person party by 2. She is working in the image making layer because she is "tied to the action or doing".

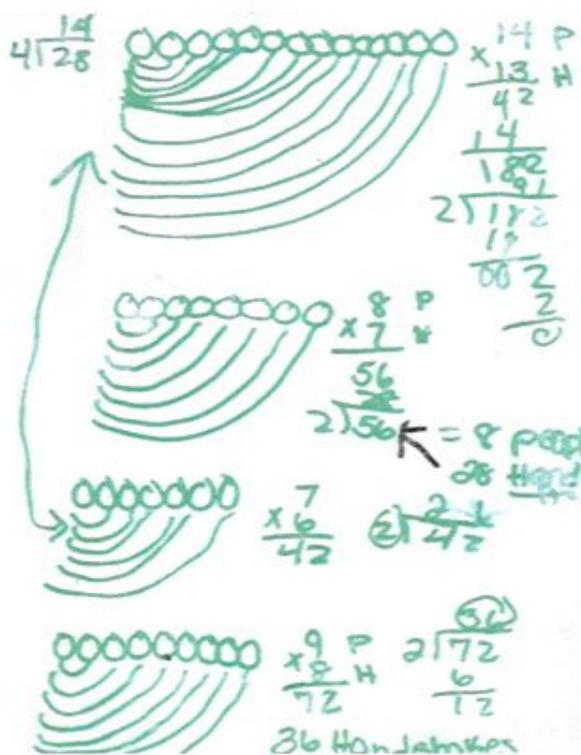


Figure 1- Aiesha's move to image making

MOVING FROM IMAGE MAKING TO IMAGE HAVING

Two weeks later the students were challenged to begin a new task involving an extension of the original task. In this episode, another group member, Bea, questions Aiesha about her initial representation. Aiesha then restructures her knowledge to generate a representation that is more understandable to her peers. In doing so, she has developed a new and ultimately more useful representation.

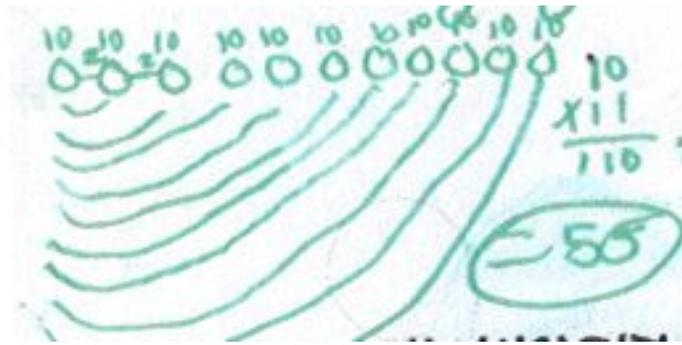


Figure 2- Aiesha’s initial strategy for finding the number of handshakes when 11 people are at the party

Bea: When you did the demonstration you did up there, I didn’t get it. [She is referring to Aiesha presenting figure 1 to the class a few days earlier.]

Aiesha: What do you mean?

Bea: All of these lines [pointing to the loops on figure 2]. What about these people [pointing to all of the circles on the right]?

Aiesha: I’m going to show you all. I am multiplying [writing $11 \times 10 = 110$].

Bea: This person [pointing to the circle all the way on the left] is shaking hands with all of these people, and this is all of his shakes. Right, and how many handshakes is right here [pointing to the first circle to the left]?

Aiesha: Ten.

Bea: And then this one (pointing to the second circle) is going to be nine, right?

Aiesha: And then eight, seven, six, five, four, three, two, one.

Aiesha then begins drawing the chart in figure 3 and explains it to her peers.

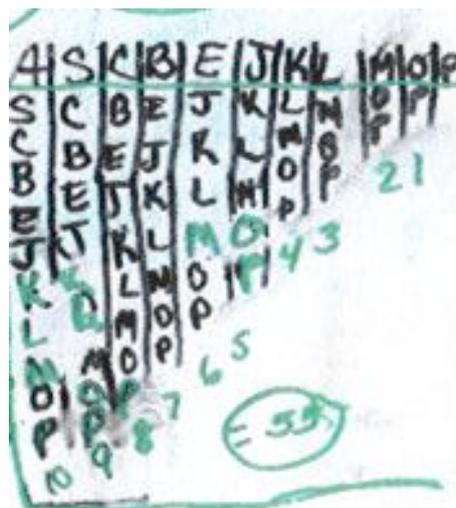


Figure 3 - Aiesha’s chart for finding the number of handshakes when 11 people are at the party

Aiesha’s explanations indicate that she has moved to the image having layer. “At the level of image having a person can use a mental construct about a topic without having to do the particular activities that brought it about.” (Pirie & Kieren, 1994, p.66) Aiesha now appears to have an “image” of the handshakes, and is no longer tied to the action of showing each handshake.

MOVING FROM IMAGE HAVING TO PROPERTY NOTICING

In this episode, two students question Aiesha's idea. This helps her to realize that her drawing shows each person shaking hands twice. Aiesha now begins to consider *why* division by 2 actually works.

Bianca: This person [pointing to the second circle on figure 2] won't shake ten people's hands. But it says every person at the party shakes hands once.

[Bianca notices the number 10 written above each circle.]

Edgar: Everyone's not going to shake everyone's hands two times.

In this case, the students' questioning helped Aiesha to notice properties about her representations, thereby prompting her movement to the property noticing layer. This layer may be characterized as one in which the individual "...can manipulate or combine aspects of his/her images to construct context specific, relevant properties." (Pirie, & Kieren, 1994) In this case, Aiesha noticed that her picture representation had double the amount of handshakes, which prompted her to build on her older representation, thereby constructing a new chart (figure 4).

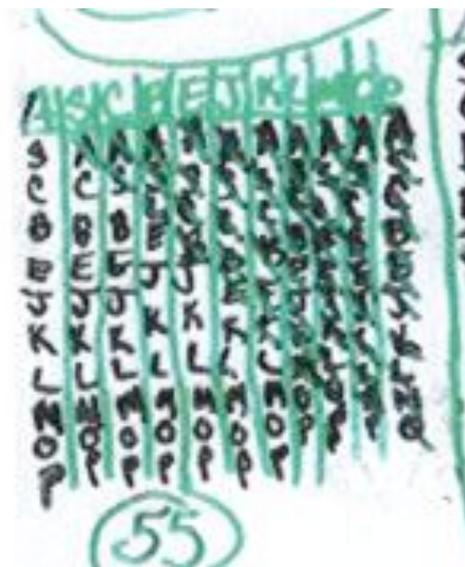


Figure 4 – Aiesha's move to property noticing

MOVING FROM PROPERTY NOTICING TO FORMALISING

After Aiesha sets up a hypothetical situation by asking Bea what she would do if there was a 500 person party, Aiesha and the other members of her group now spontaneously attempt to generate a more generalized symbolic representation that could work for any number of people or handshakes. For this, they revert back to the original problem. Aiesha draws a chart (similar to figure 4) for an eight person party and constructs a number sentence along with a formula using both words and standard algebraic notation. Ultimately, Aiesha is able to come up with the formula $[n(n-1)]/2$, which she presents to the class.

We suggest this movement to a symbolic representation moves Aiesha to the formalising layer, creating a general statement, in which a method or common quality from the previous image is abstracted (Pirie, and Kieren, 1994).

Aiesha: N equals the number of people at the party. What I did was n times, well, we're going to do n times n minus one, n minus one in parentheses [tracing the parentheses with her marker on figure 5]. First what we have to do, eight, there's eight people, we have to take minus one, so there's seven [writing $8-1 = 7$ on figure 5]. So, n times n minus one, then you divide that by two. You would multiply eight by seven, then you would divide that whole answer by two.

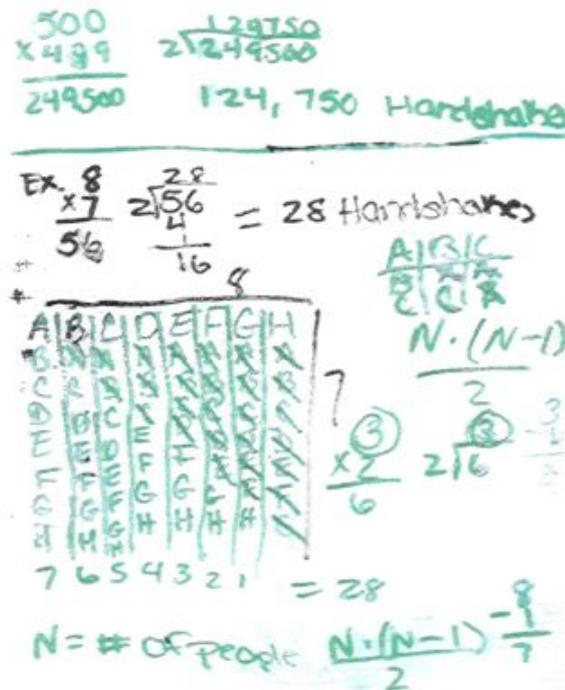


Figure 5: Working in the formalising layer

As Aiesha was presenting her formula, another student, Shaniqua, questioned her and set up a hypothetical situation based on the existing problem. Aiesha showed that her idea was valid, using multiple representations to solve the hypothetical situation (words, numbers, symbols, a chart, picture representation and acting it out), and was questioned into linking these representations to each other.

Shaniqua: [Shaniqua raises her hand during Aiesha's presentation. Aiesha calls on her.] I disagree with something. She said that there was five hundred people at the party and each of those people shake hands with four hundred and ninety nine people's hands [initially directing the comment to the teacher/researcher]. That's not true because if you do that, then you're saying each person shook [now directing the comment to Aiesha]... Ok, let's say there is three people at the party...

Aiesha: Yeah.

Shaniqua: And you are saying that every one of these three people are shaking the same three people's hands? They are shaking the same people's hands?

Aiesha: Do you want to see how that works with three people?

Shaniqua: Yeah.

Aiesha describes how to use the formula for three people.

Luis: What's the n?

Aiesha: All right, the n equals the number of people, n, three people, right. What you have to do first is 3 minus one, it gives you two. Then you have to do three times two, and it gives you six. You divide two into six and it gives you three. That's how many handshakes.

Aiesha draws the chart for 3 people at the party (see figure 5) and explains it.

Crystal: Why do you use the number two to divide?

Aiesha: All right, I use two because look, when two people (shaking Bianca's hand), it gives you two handshakes (pointing to her and Bianca), but normally...

Aiesha explains that she is initially counting both handshakes, then dividing the second handshake out. She draws the chart (bottom of figure 6) as if she were A and Shaniqua were B. She continues by writing a 2 between two circles (which represent people) on her picture to show the two handshakes that took place between each set of two people (top of figure 6 and the top of figure 2) to answer Crystal's question.

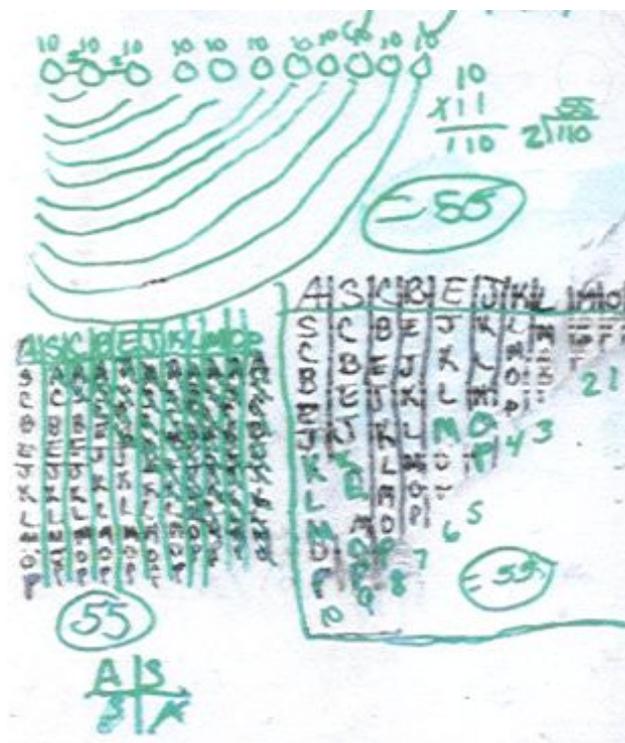


Figure 6: Linking representations to each other

Aiesha was able to show how her formula mapped into her original representation involving circles and loops, the action of actually shaking hands, as well as her chart with letters. Her ability to set up a hypothetical situation about the existing problem, develop multiple representations for the same idea, connect the representations to each other, and ultimately provide a solution that is generalizable indicates that she has reached the formalising layer. Further, some six months later, Aiesha's class was given the opportunity to investigate a task that was structurally similar to this handshake problem. Within a few minutes, Aiesha and her group moved through most of the representations they constructed six months earlier, and reconstructed the formula to generalize, using the correct symbolic notation. Interestingly enough, many students around the room also used the formula Aiesha presented six months earlier for this new task.

CONCLUSIONS

We conclude that student-to-student interactions and questions played a central role in Aiesha's movement from primitive knowing to formalising, as well as her movement to linking representations to each other. This ultimately led to her ability to retain and retrieve her ideas when presented with similar types of problems months later, which is a central goal of the teaching and learning process. Of course, we cannot say with complete certainty that these interactions were exclusively responsible for the development of the ideas, however, we believe that our analysis suggests that they played a key role. While it is not possible to draw overwhelming conclusions based on this limited example, we do believe that an analysis of

this type has the potential to call attention to the importance of providing meaningful opportunities for such student-to-student interaction.

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